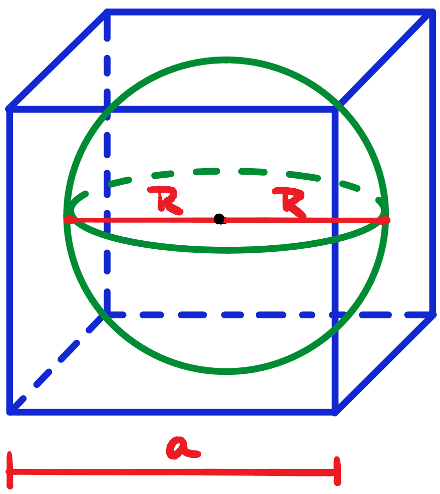
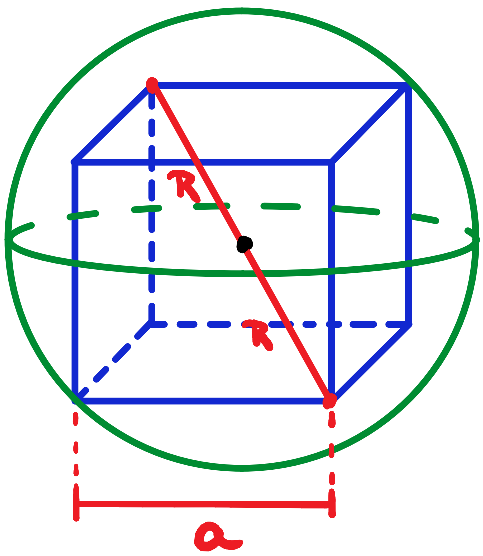
INSCRIÇÃO DE SÓLIDOS

CUBO E ESFERA



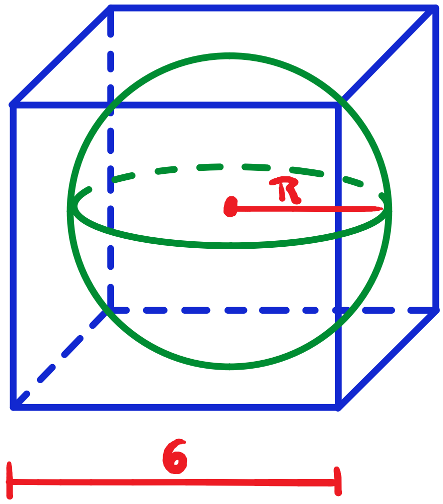
DIÂM = LADO

$2R = a$



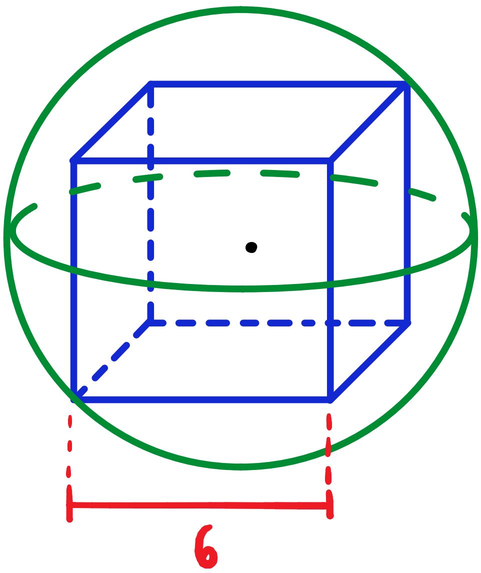
DIÂM = DIAG

$2R = a\sqrt{3}$



$$2R = 6$$

$$\underline{R = 3}$$

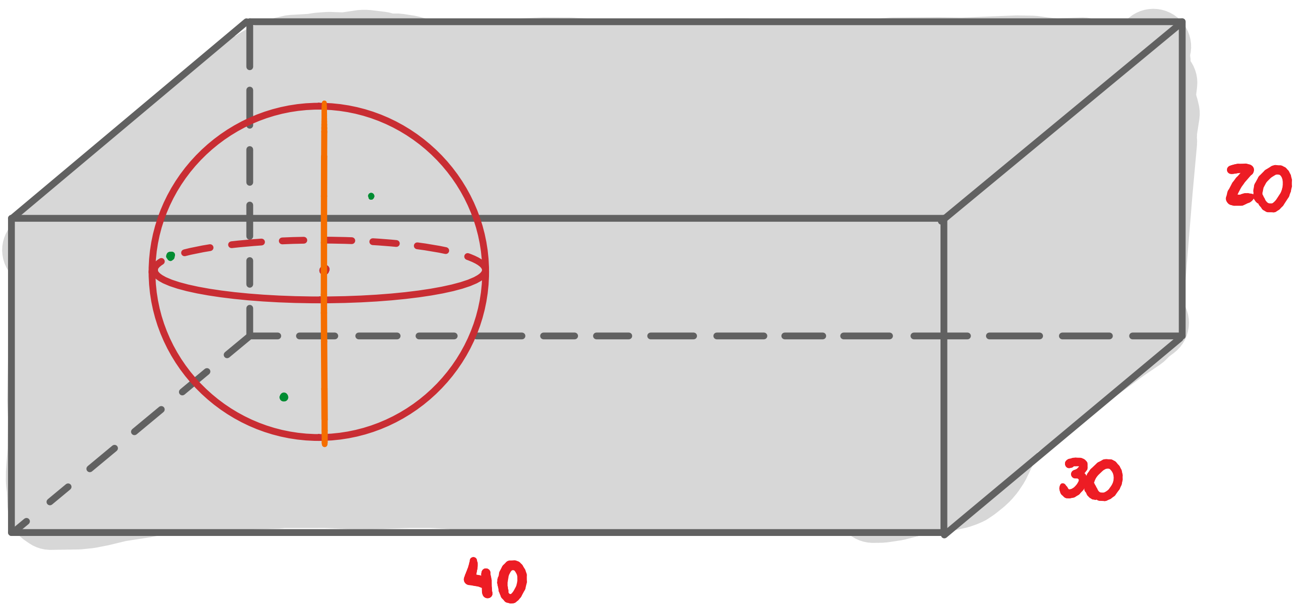


$$2R = 6 \cdot \sqrt{3}$$

$$\underline{R = 3\sqrt{3}}$$

EXEMPLO

CALCULE O RAIO DA MAIOR ESFERA QUE PODE SER INSCRITA EM UM PARALELEPÍPEDO RETO RETÂNGULO DE LADOS 20, 30 E 40.

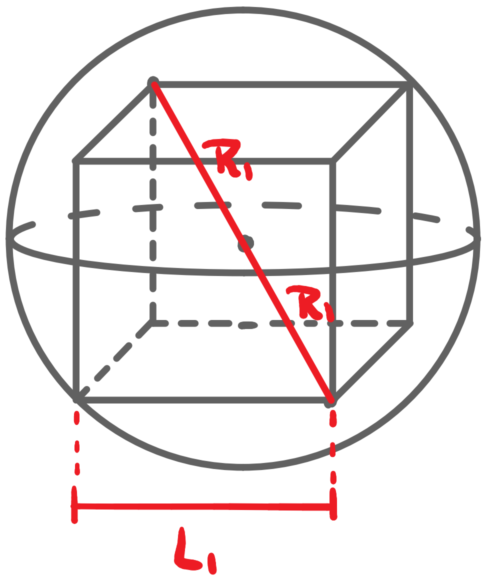


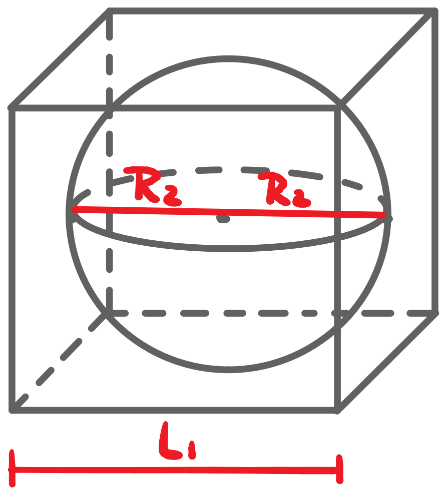
$$2R = 20$$

$$R = 10$$

EXEMPLO

SEJA UMA ESFERA DE RAIO 1. UM CUBO É INSCRITO NESSA ESFERA. POR USA VEZ, UMA ESFERA É INSCRITA NESSE CUBO. E ASSIM POR DIANTE. CALCULE A SOMA DAS ÁREAS DAS INFINITAS ESFERAS.





$$L_1\sqrt{3} = 2R_1$$

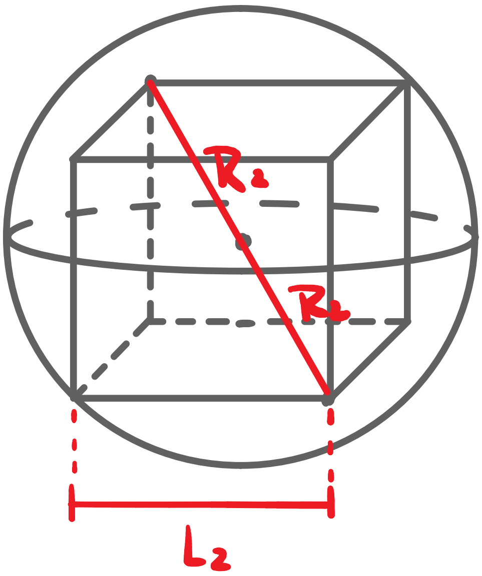
$$L_1 = \frac{2R_1}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}}$$

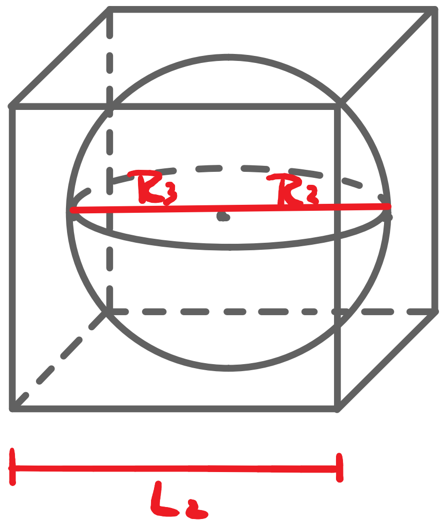
$$L_1 = \frac{2R_1\sqrt{3}}{3}$$

$$2R_2 = L_1$$

$$R_2 = \frac{L_1}{2}$$

$$\boxed{R_2 = \frac{R_1\sqrt{3}}{3}}$$





$$L_2 = \frac{2R_2\sqrt{3}}{3}$$

$$R_3 = \frac{L_2}{2}$$

$$\boxed{R_3 = \frac{R_2\sqrt{3}}{3}}$$

$$\text{RAIOS}\left\{ R_1,\ R_1\frac{\sqrt{3}}{3},\ R_1\left(\frac{\sqrt{3}}{3}\right)^2,\ R_1\left(\frac{\sqrt{3}}{3}\right)^3,\ \ldots \right\}$$

$$\text{PG.} \quad q = \frac{\sqrt{3}}{3}$$

$$q^2 = \left(\frac{\sqrt{3}}{3}\right)^2 = \frac{1}{3}$$

$$A_1 = 4\pi R_1^2$$

$$A_2 = 4\pi R_2^2 = 4\pi\left(\frac{R_1\sqrt{3}}{3}\right)^2 = \frac{4\pi R_1^2}{3}$$

$$A_3 = \frac{4\pi R_1^2}{9}$$

SOMA DAS ÁREAS:

$$4\pi R_1^2 + \frac{1}{3}\cdot 4\pi R_1^2 + \frac{1}{9}\cdot 4\pi R_1^2 + \dots$$

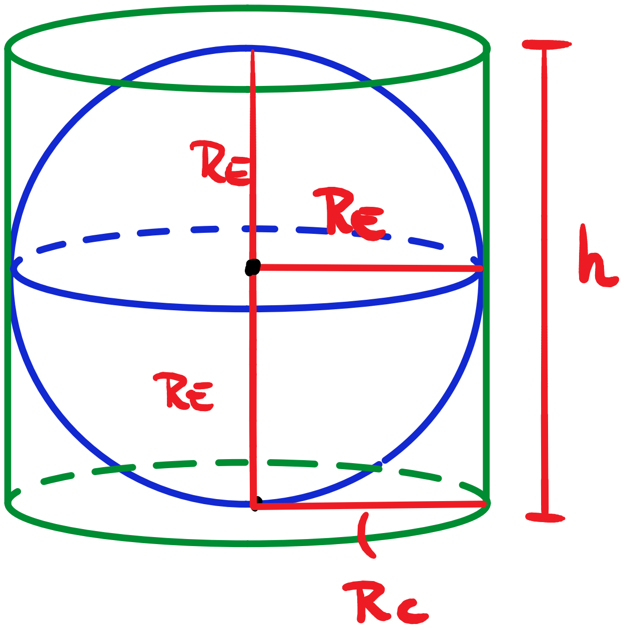
$$S_\infty = \frac{a_1}{1-q}$$

$$S_\infty = \frac{4\pi R_1^2}{1-1/3} = \frac{4\pi R_1^2}{2/3}$$

$$= \overset{2}{\cancel{4}}\pi R_1^2 \cdot \frac{3}{\cancel{2}} = 6\pi R_1^2$$

$$R_1 = 1 \longrightarrow \underline{S_\infty = 6\pi}$$

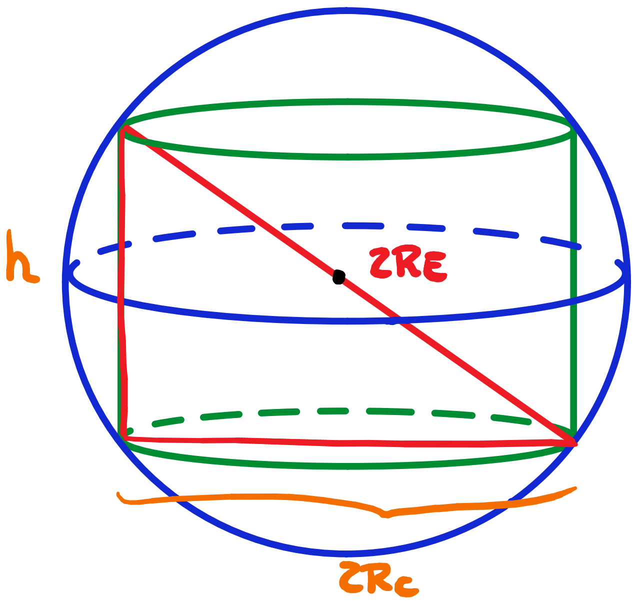
Esfera e Cilindro



$R_E = R_C$

$h = 2R_E$

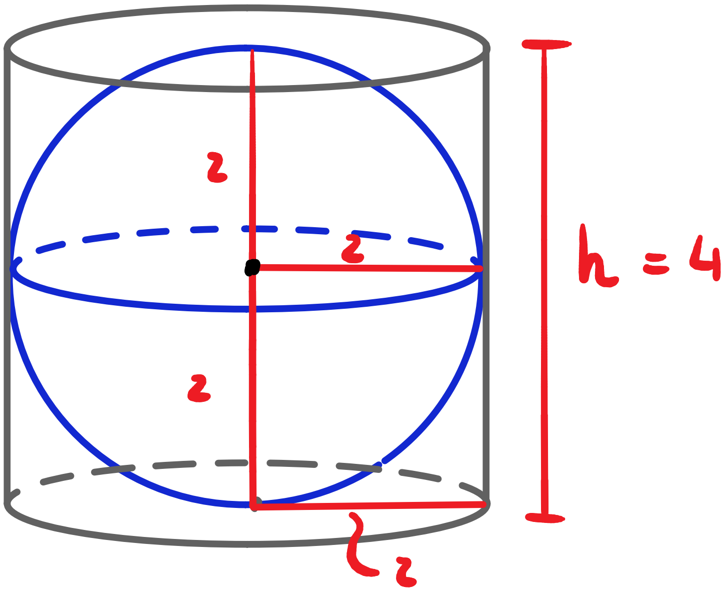
* Cil. Equilátero



$$\boxed{(2R_E)^2 = (2R_C)^2 + h^2}$$

EXEMPLO

UMA ESFERA ESTÁ INSCRITA EM UM CILINDRO. SE O RAIO DESSA ESFERA É 2, CALCULE O VOLUME DO CILINDRO.



$R_c = 2$

$h = 4$

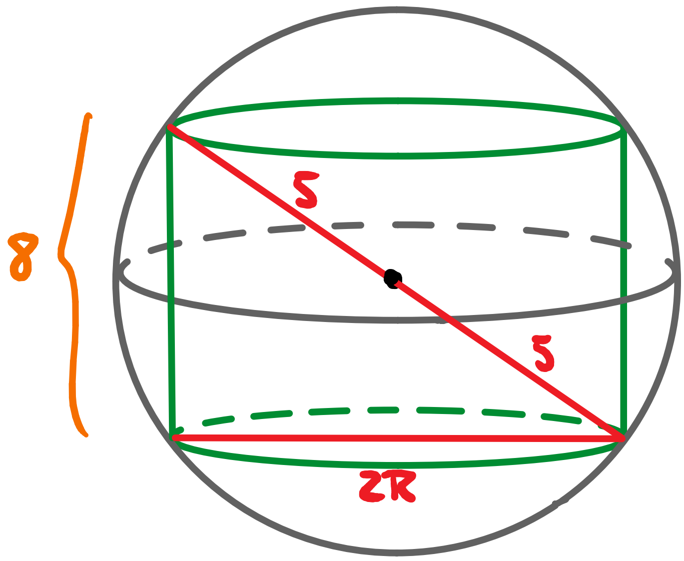
$$V = \pi R^2 h$$

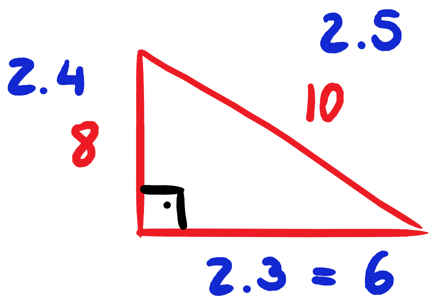
$$= \pi \cdot 2^2 \cdot 4$$

$$\underline{V = 16\pi}$$

EXEMPLO

UM CILINDRO DE ALTURA 8 ESTÁ INSCRITO EM UMA ESFERA DE RAIO 5. CALCULE O VOLUME DO CILINDRO.





$2R = 6$

$\underline{R = 3}$

$V = \pi R^2 h$

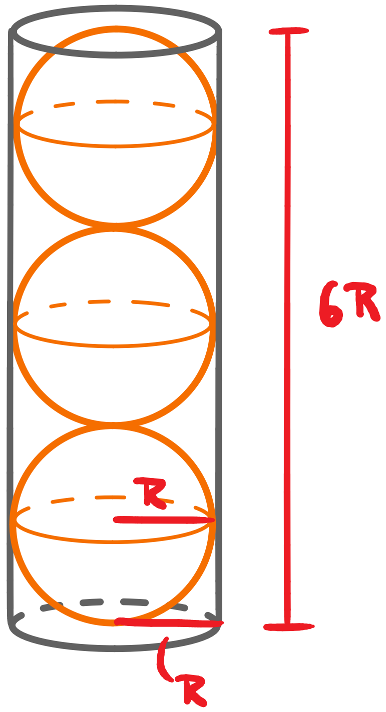
$V = \pi \cdot 3^2 \cdot 8$

$\underline{V = 72\pi}$

EXEMPLO

TRÊS ESFERAS ESTÃO INSCRITAS EM UM CILINDRO COMO MOSTRA A FIGURA.

CALCULE A FRAÇÃO DO VOLUME DO CILINDRO QUE SE ENCONTRA VAZIO.



$$V_{TOTAL} = \pi R^2 \cdot 6R$$
$$= 6\pi R^3$$

$$V_{CHEIO} = \cancel{3} \cdot \frac{4}{\cancel{3}} \pi R^3$$
$$= 4\pi R^3$$

FR. CHEIA : $\dfrac{4\cancel{\pi}\cancel{R^3}}{6\cancel{\pi}\cancel{R^3}} = \dfrac{2}{3}$

FR. VAZIA : $\dfrac{1}{3}$

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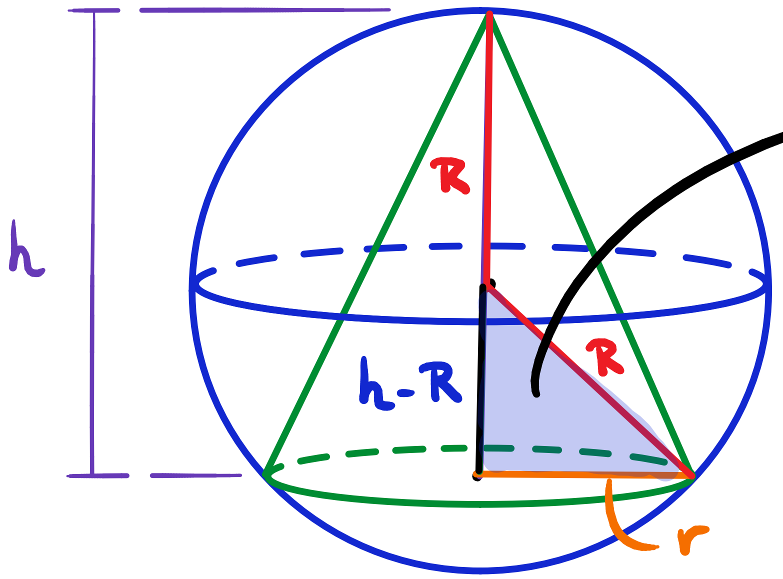
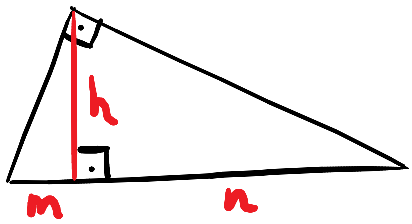
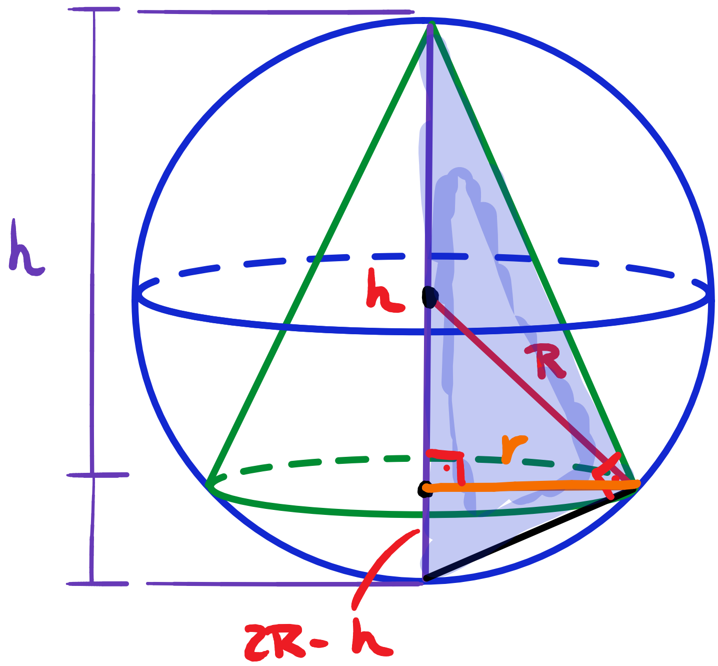
Esfera e Cone



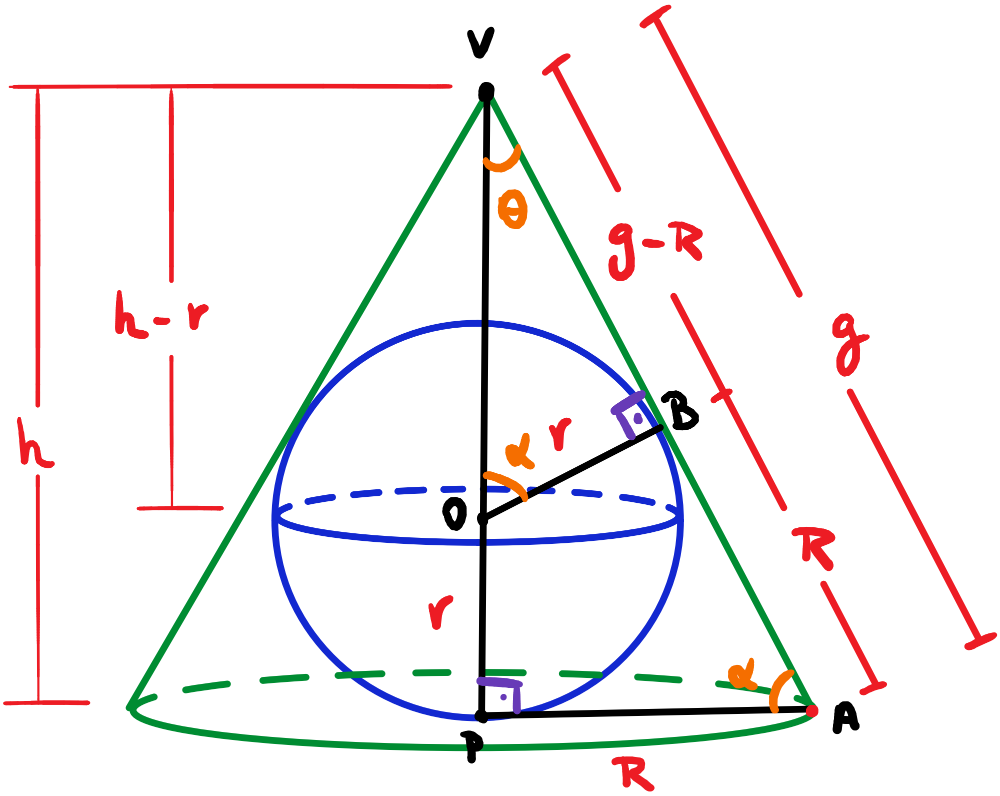
$$R^2 = r^2 + (h - R)^2$$



$$h^2 = mn$$



$$r^2 = h(2R - h)$$



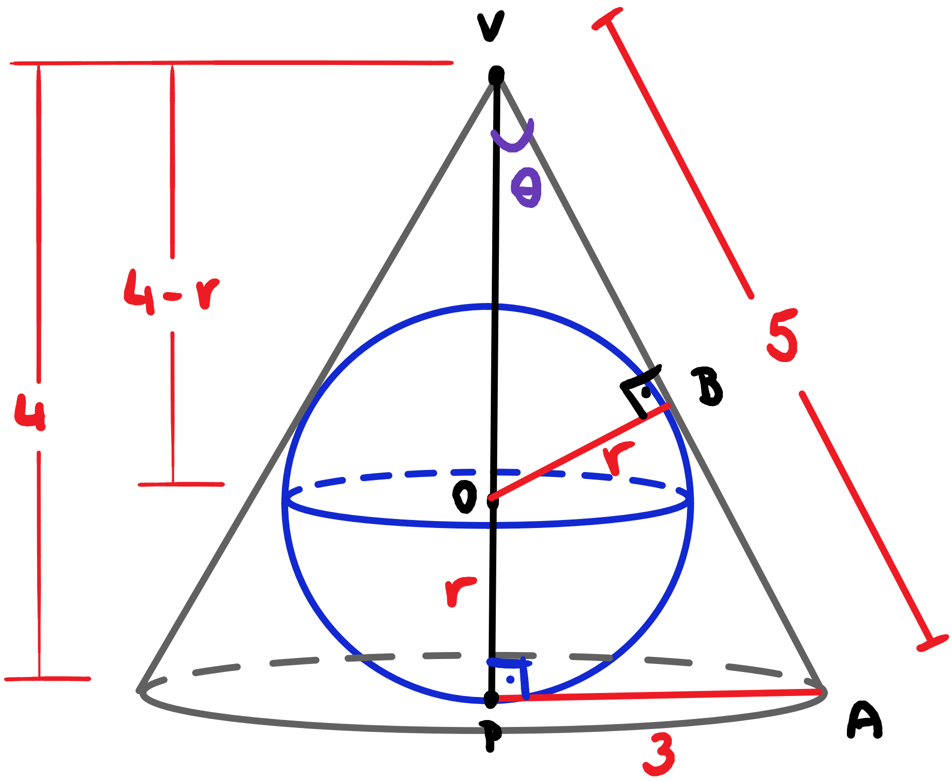
$$\Delta VBO \sim \Delta VPA$$

$$\boxed{\frac{OB}{PA} = \frac{VB}{VP} = \frac{VO}{VA}}$$

$$\boxed{\frac{r}{R} = \frac{g-R}{h} = \frac{h-r}{g}}$$

EXEMPLO

CALCULE O VOLUME DA ESFERA INCRITA EM UM CONE RETO DE RAIO DA BASE 3 E ALTURA 4.



$\Delta VBO \sim \Delta VPA$

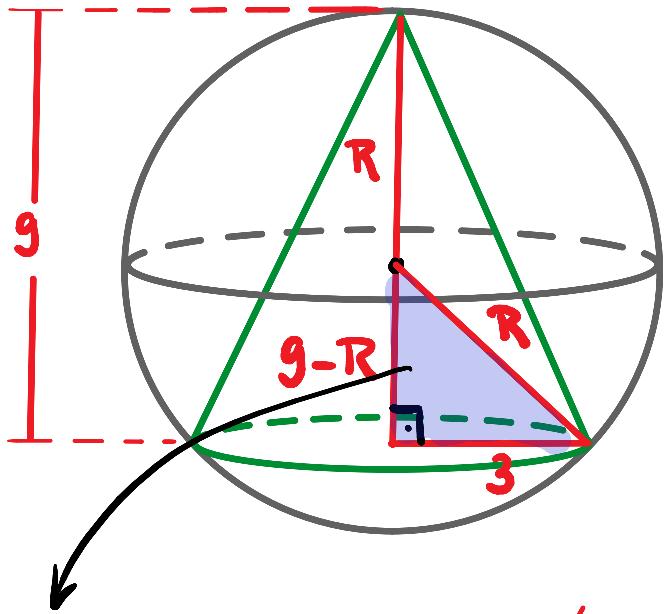
$$\frac{r}{3} = \frac{4-r}{5} \rightarrow 5r = 12 - 3r$$

$$8r = 12$$

$$\boxed{r = \frac{3}{2}}$$

EXEMPLO

UM CONE RETO DE RAIO DA BASE 3 E ALTURA 9 ESTÁ INSCRITO EM UMA ESFERA. CALCULE O VOLUME DA ESFERA.



$$R^2 = (9-R)^2 + 3^2 \rightarrow \cancel{R^2} = 81 - 18R + \cancel{R^2} + 9$$

$$18R = 90 \rightarrow \underline{R = 5}$$

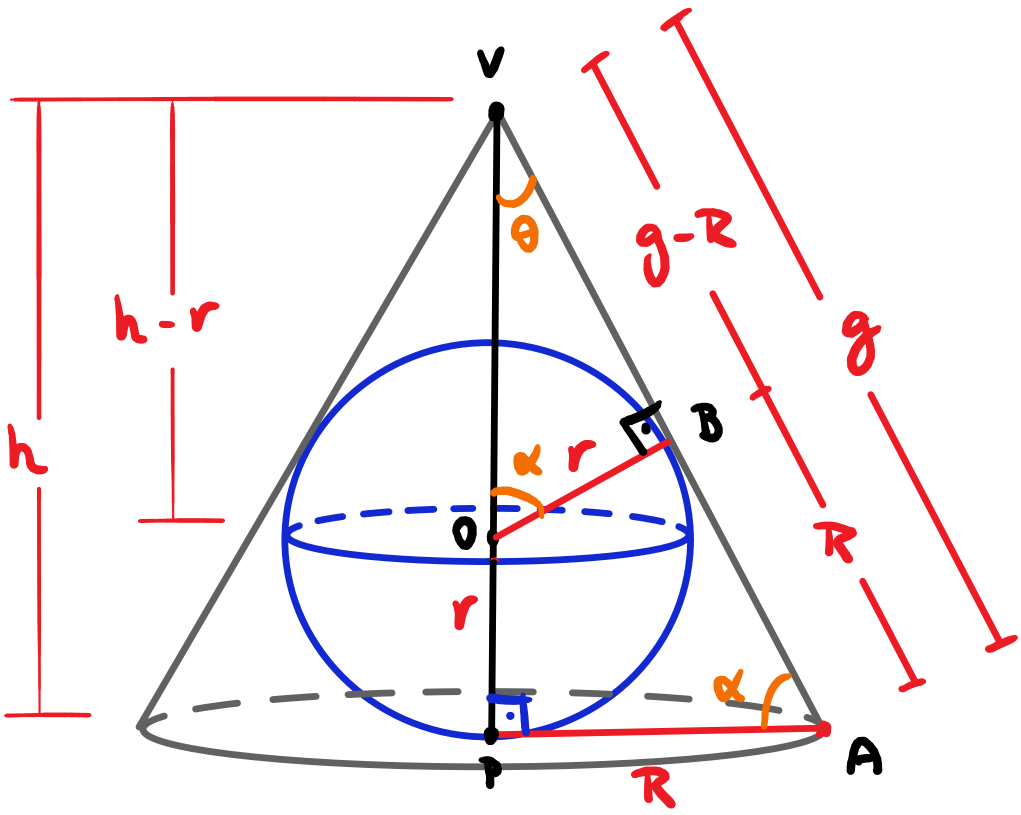
$$V = \frac{4}{3}\pi \cdot 5^3 \rightarrow \boxed{V = \frac{500\pi}{3}}$$

EXEMPLO

SEJA UMA ESFERA INSCRITA EM UM CONE RETO.

A RAZÃO ENTRE A ÁREA SUPERFICIAL DA ESFERA E A ÁREA LATERAL DO CONE É 2/3.

CALCULE O ÂNGULO FORMADO ENTRE A ALTURA E A GERATRIZ DO CONE.



$$\frac{r}{R} = \frac{g-R}{h} \quad ; \quad g^2 = R^2 + h^2$$

$$h = \sqrt{g^2 - R^2}$$

$$\frac{A_{ESF}}{A_{LAT}} = \frac{2}{3} \rightarrow \frac{4\cancel{\pi} r^2}{\cancel{\pi} Rg} = \frac{2}{3}$$

$$\boxed{\frac{r^2}{Rg} = \frac{1}{6}}$$

$$\frac{r}{R} = \frac{g-R}{h} \rightarrow \left(\frac{r}{R}\right)^2 = \left(\frac{g-R}{\sqrt{g^2-R^2}}\right)^2$$

$$\frac{r^2}{R^2} = \frac{(g-R)^2}{g^2-R^2}$$

$$\frac{r^2}{R^2} = \frac{(g-R)^{\cancel{2}}}{(g+R)\cancel{(g-R)}}$$

$$\frac{r^2}{R^2} = \frac{g-R}{g+R}$$

$$\frac{\cancel{R}g}{6.R^{\cancel{2}}} = \frac{g-R}{g+R}$$

$$g^2 + gR = 6Rg - 6R^2$$

$$\frac{g^2}{R^2} - \frac{5Rg}{R^2} + \frac{6R^2}{R^2} = \frac{0}{R^2} \qquad \operatorname{sen}\theta = \frac{R}{g}$$

$$\left(\frac{g}{R}\right)^2 - 5\left(\frac{g}{R}\right) + 6 = 0 \quad ; \quad \frac{g}{R} = x$$

$$x^2 - 5x + 6 = 0$$

$$\frac{g}{R} = 2 \quad \text{ou} \quad \frac{g}{R} = 3$$

$$\frac{R}{g} = \frac{1}{2} \qquad \frac{R}{g} = \frac{1}{3}$$

$$\operatorname{sen}\theta = \frac{1}{2} \qquad \operatorname{sen}\theta = 1/3$$

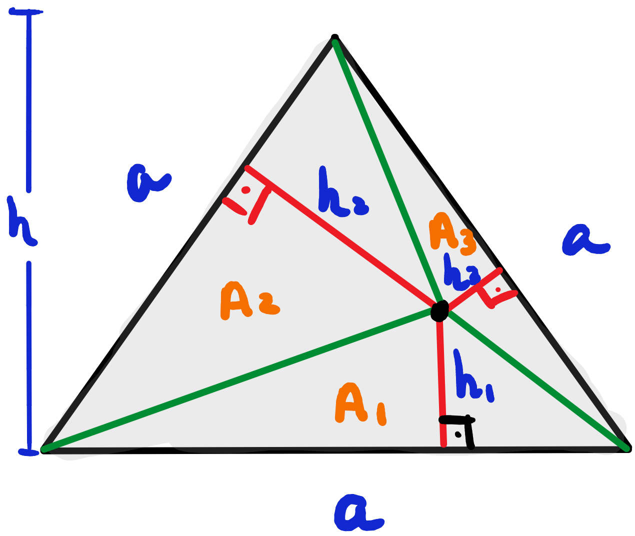
$$\theta = 30^\circ \qquad \theta = \operatorname{arcsen}\left(\frac{1}{3}\right)$$

LEMBRANDO ...

TEOREMA - TRIÂNGULO EQUILÁTERO

SEJA UM PONTO INTERIOR A UM TRIÂNGULO EQUILÁTERO.

A SOMA DAS DISTÂNCIAS DESSE PONTO AOS LADOS DO TRIÂNGULO É IGUAL À ALTURA DESSE TRIÂNGULO.



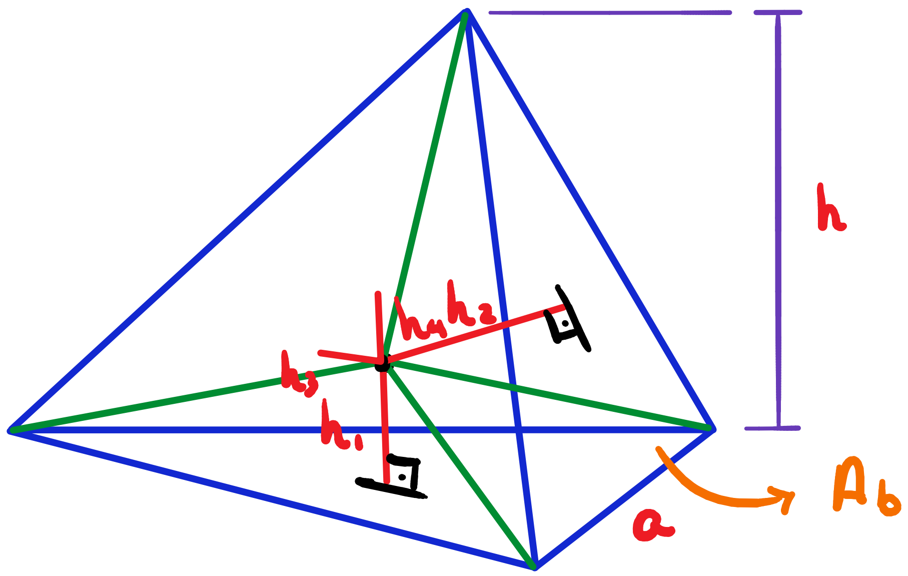
$$A_T = A_1 + A_2 + A_3$$

$$\frac{\cancel{a}.h}{\cancel{2}} = \frac{\cancel{a}.h_1}{\cancel{2}} + \frac{\cancel{a}.h_2}{\cancel{2}} + \frac{\cancel{a}.h_3}{\cancel{2}}$$

$$\boxed{h = h_1 + h_2 + h_3}$$

Teorema – Tetraedro Regular

A soma das distâncias de um ponto interno a um tetraedro regular às suas faces é igual à altura desse tetraedro.



$$V_T = V_1 + V_2 + V_3 + V_4$$

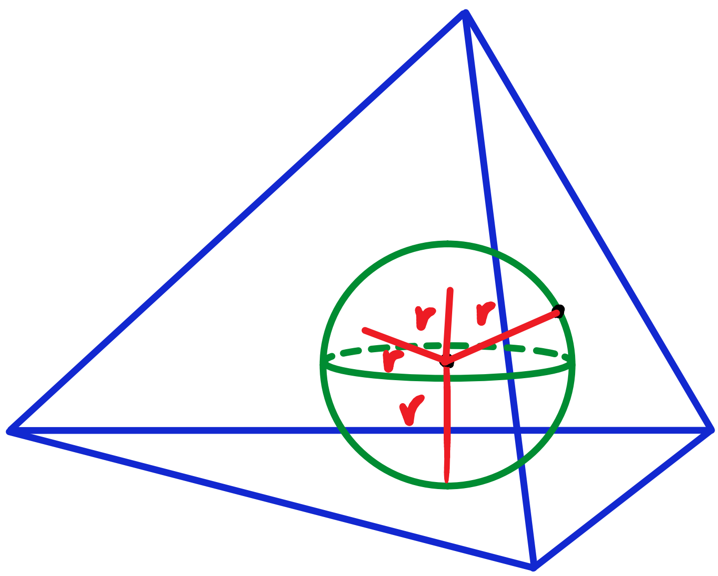
$$\frac{1}{\cancel{3}} \cdot \cancel{A_b} \cdot h = \frac{1}{\cancel{3}} \cancel{A_b} \cdot h_1 + \frac{1}{\cancel{3}} \cancel{A_b} \cdot h_2 + \frac{1}{\cancel{3}} \cancel{A_b} \cdot h_3 + \frac{1}{\cancel{3}} \cancel{A_b} \cdot h_4$$

$$\boxed{h = h_1 + h_2 + h_3 + h_4}$$

TETRAEDRO REGULAR E ESFERA

ESFERA INSCRITA

CENTRO EQUIDISTANTE DAS FACES.

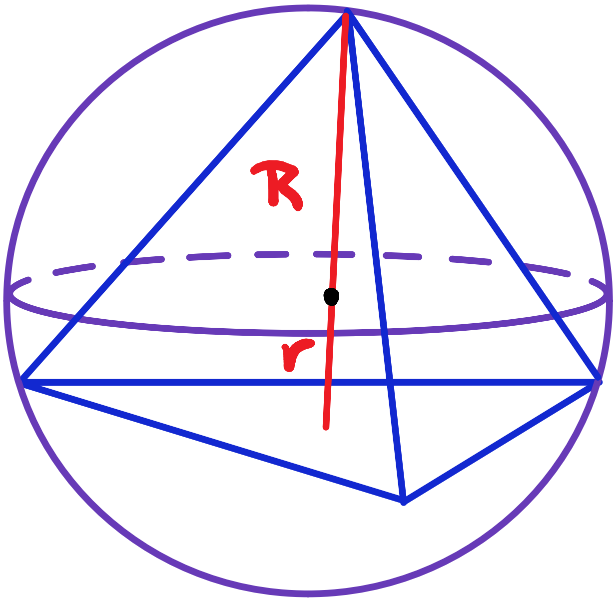


$$r + r + r + r = h$$

$$4r = h$$

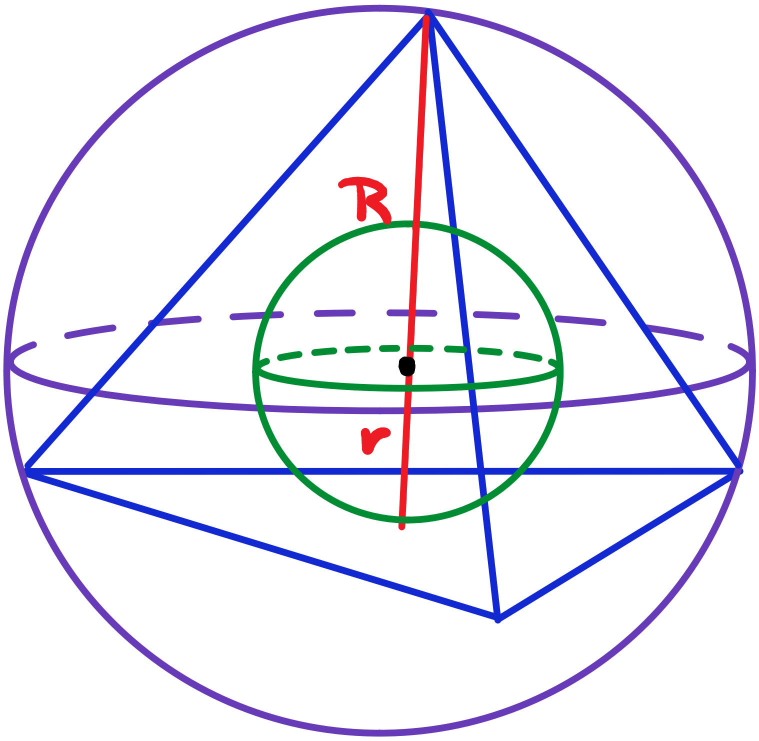
$$\boxed{r = \frac{h}{4}}$$

Esfera Circunscrita



$$\boxed{R = \frac{3}{4} h}$$

Resumo

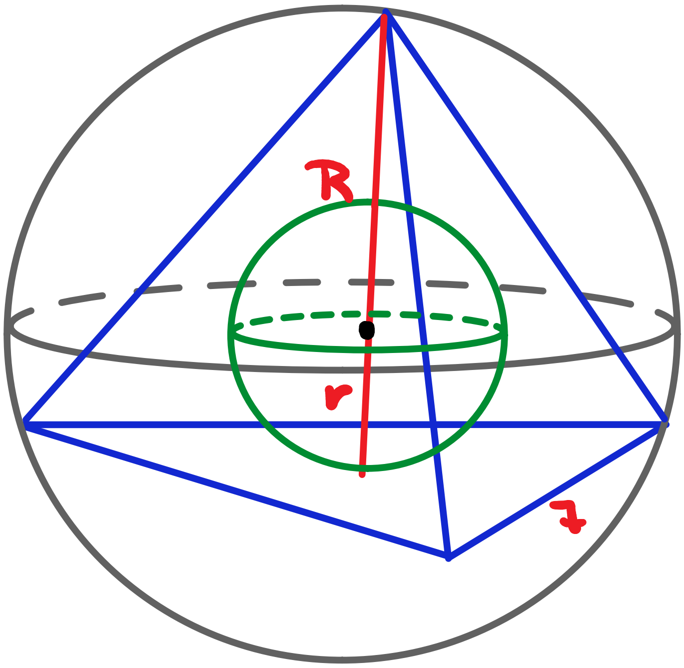


$$\boxed{r = \frac{1}{4}h}$$

$$\boxed{R = \frac{3}{4}h}$$

EXEMPLO

SEJA UM TETRAEDRO DE LADO 7. CALCULE OS RAIOS DAS ESFERAS INCRITA E CIRCUNSCRITA A ESSE TETRAEDRO.



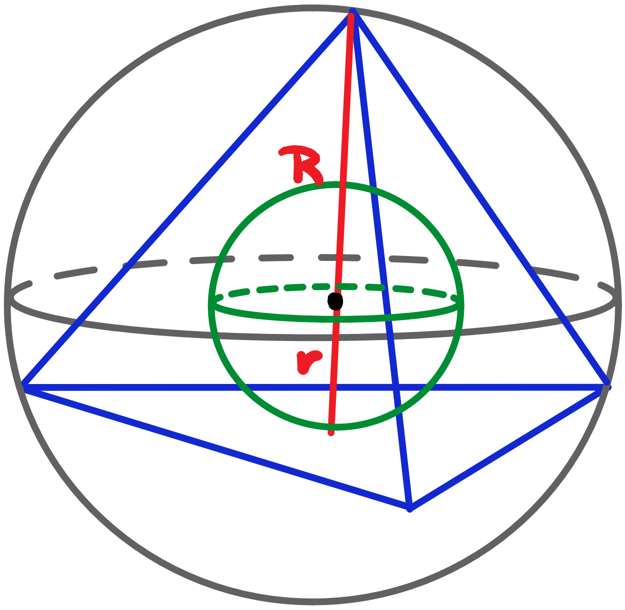
$$h = \frac{a\sqrt{6}}{3} \rightarrow \boxed{h = \frac{7\sqrt{6}}{3}}$$

$$r = \frac{1}{4} \cdot h \rightarrow r = \frac{7\sqrt{6}}{12}$$

$$R = 3 \cdot r \rightarrow R = \cancel{3} \cdot \frac{7\sqrt{6}}{\cancel{12}\,4} \rightarrow \boxed{R = \frac{7\sqrt{6}}{4}}$$

EXEMPLO

O RAIO DA ESFERA INSCRITA A UM TETRAEDRO REGULAR É 3. CALCULE O VOLUME DA ESFERA CIRCUNSCRITA A ESSE TETRAEDRO.



$$r = 3$$

$$R = 3 \cdot r$$

$$R = 3 \cdot 3$$

$$\underline{R = 9}$$

$$V = \frac{4}{3} \cdot \pi R^3$$

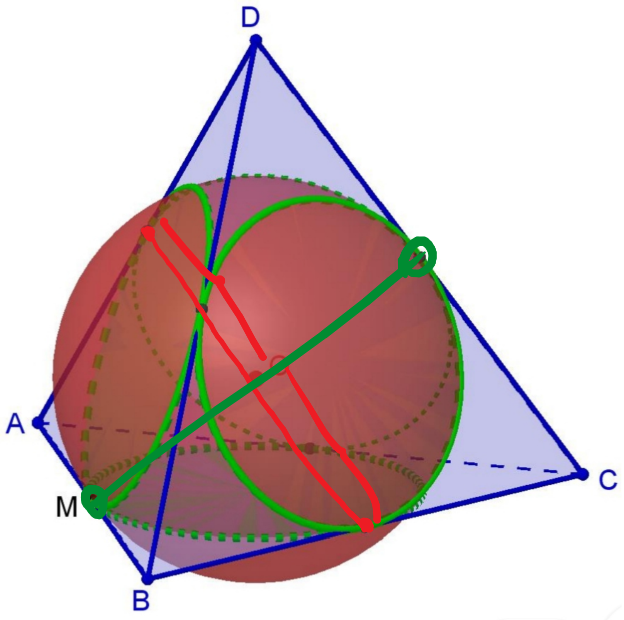
$$V = \frac{4}{\cancel{3}} \cdot \pi \cdot \overset{3}{\cancel{9}} \cdot 9 \cdot 9$$

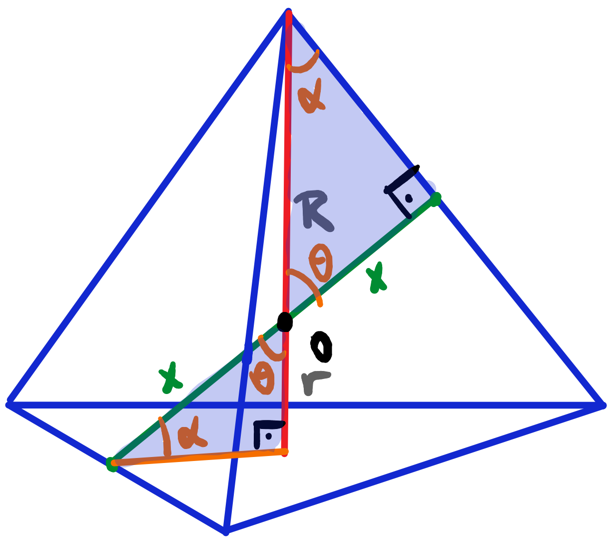
$$\begin{array}{r} {}^{1\,1} \\ 243 \\ \times\ 4 \\ \hline 972 \end{array}$$

$$\underline{V = 972\pi}$$

EXEMPLO

CALCULE O RAIO DA ESFERA TANGENTE ÀS ARESTAS DE UM TETRAEDRO REGULAR DE LADO a.





SEMELHANÇA : $\frac{x}{R} = \frac{r}{x}$

$$x^2 = R \cdot r$$

$$R = \frac{3}{4} h = \frac{3}{4} \cdot \frac{a\sqrt{6}}{3} = \frac{a\sqrt{6}}{4}$$

$$r = \frac{1}{4} h = \frac{1}{4} \cdot \frac{a\sqrt{6}}{3} = \frac{a\sqrt{6}}{12}$$

$$x^2 = R . r$$

$$x^2 = \frac{a\sqrt{6}}{4} \cdot \frac{a\sqrt{6}}{4} \cdot \frac{1}{3}$$

$$x^2 = \left(\frac{a\sqrt{6}}{4}\right)^2 \cdot \frac{1}{3}$$

$$x = \frac{a\sqrt{6}}{4} \cdot \frac{1}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}}$$

$$x = \frac{a\sqrt{\cancel{3}} . \sqrt{2} . \sqrt{\cancel{3}}}{4 . \cancel{3}} \rightarrow \boxed{x = \frac{a\sqrt{2}}{4}}$$

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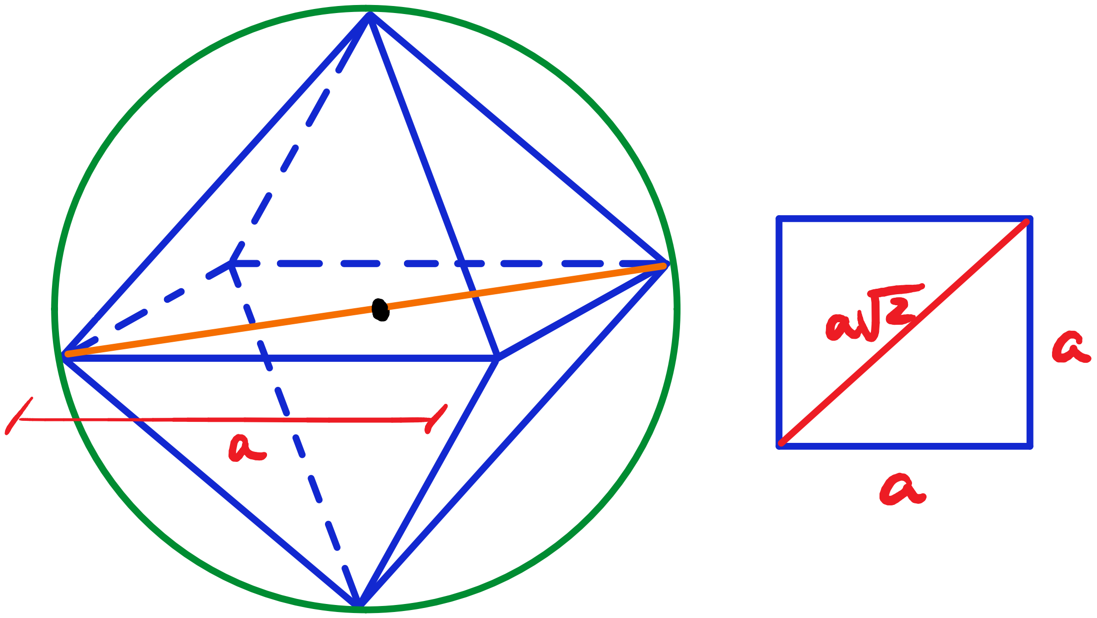
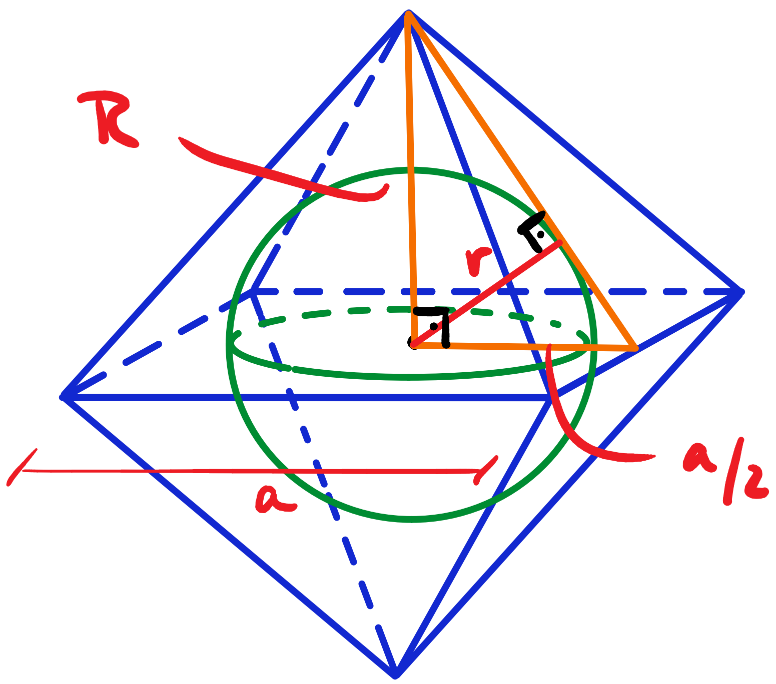
Octaedro e Esfera

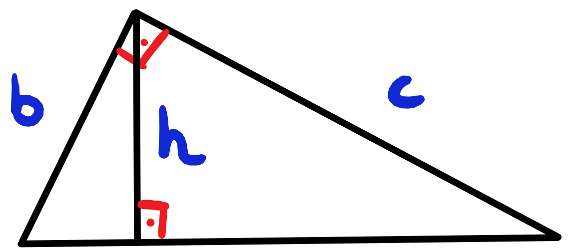


$$\boxed{2R = a\sqrt{2}}$$

$$R = \frac{a\sqrt{2}}{2}$$

$$R^2 = \frac{a^2 \cdot 2}{4} = \frac{a^2}{2}$$





$$\rightarrow \quad \frac{1}{h^2} = \frac{1}{b^2} + \frac{1}{c^2}$$

$$\frac{1}{r^2} = \frac{1}{(a/2)^2} + \frac{1}{R^2} = \frac{4}{a^2} + \frac{2}{a^2}$$

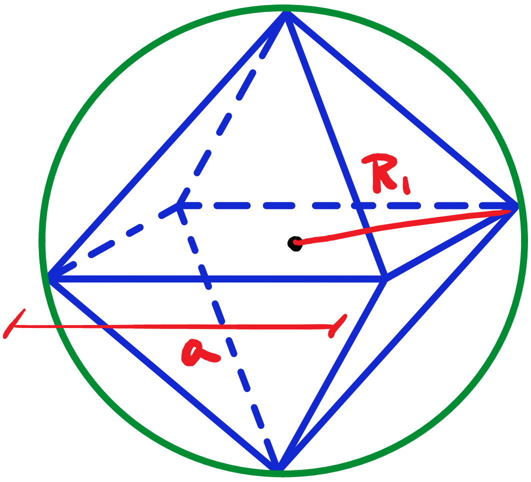
$$\frac{1}{r^2} = \frac{8}{a^2} \rightarrow r^2 = \frac{a^2}{8} \rightarrow r = \frac{a}{2\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}}$$

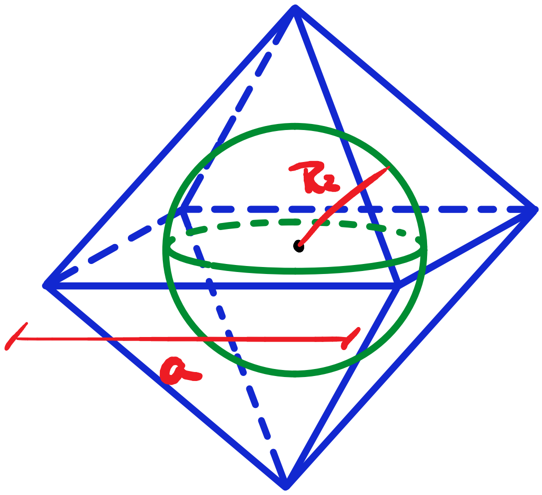
$$\boxed{r = \frac{a\sqrt{2}}{4}}$$

EXEMPLO

UM OCTAEDRO REGULAR ESTÁ INSCRITO EM UMA ESFERA DE RAIO 1. NESSE OCTAEDRO, INSCREVE-SE UMA NOVA ESFERA. NESSA ESFERA, INSCREVE-SE UM NOVO OCTAEDRO. E ASSIM POR DIANTE.

CALCULE A SOMA DOS VOLUMES DESSAS INFINITAS ESFERAS.





$$2R_1 = a\sqrt{2}$$

$$\frac{\cancel{\sqrt{2}}\cdot\sqrt{2}\cdot R_1}{\cancel{\sqrt{2}}} = a$$

$$a = R_1\sqrt{2}$$

$$R_2 = \frac{a\sqrt{2}}{4}$$

$$R_2 = \frac{R_1\sqrt{2}\cdot\sqrt{2}}{4}$$

$$\boxed{R_2 = \frac{R_1}{2}}$$

$$R_3 = \frac{R_2}{2} \rightarrow \boxed{R_3 = \frac{R_1}{4}}$$

$$R_4 = \frac{R_1}{8}, \ \ldots$$

$(R_1, R_2, R_3, \ldots) \rightarrow$ PG. RAZÃO $\frac{1}{2}$

$V_1 = \frac{4}{3}\pi R_1^3$; $V_2 = \frac{4}{3}\pi \left(\frac{R_1}{2}\right)^3$

$V_2 = \frac{1}{8} \cdot V_1$ $\quad \left(\frac{1}{2}\right)^3$

$(V_1, V_2, V_3, \ldots) \rightarrow$ PG. RAZÃO $\frac{1}{8}$

$$\text{SOMA} = \frac{V_1}{1-q} = \frac{V_1}{1-\frac{1}{8}} = \frac{8}{7} V_1$$

$$\text{SOMA} = \frac{8}{7} \cdot \frac{4}{3} \pi \cdot 1^3$$

$$\boxed{\text{SOMA} = \frac{32\pi}{21}}$$

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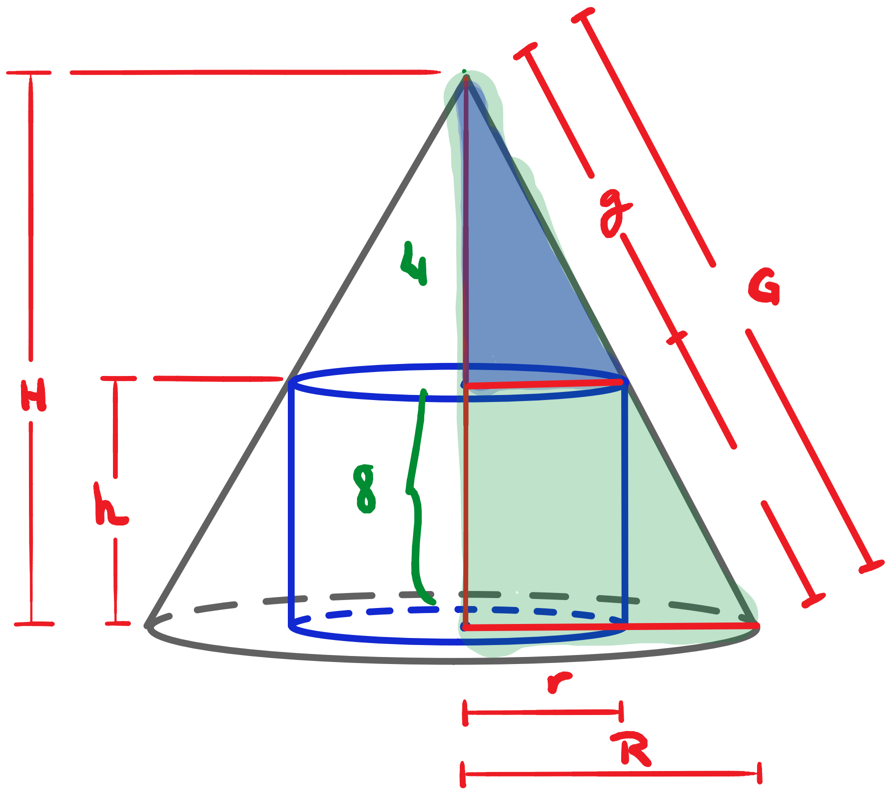
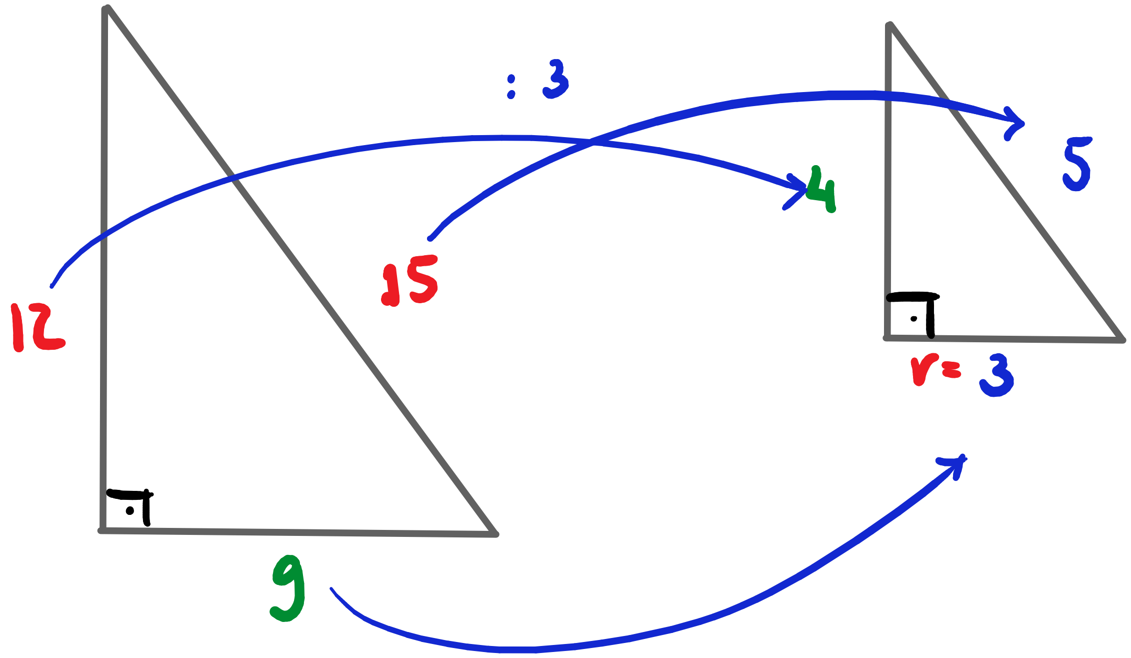
EXEMPLO

SEJA UM CONE RETO DE ALTURA 12 E GERATRIZ 15. CALCULE O VOLUME DO CILINDRO DE ALTURA 8 INSCRITO NESSE CONE.



TRIÂNGULOS SEMELHANTES :



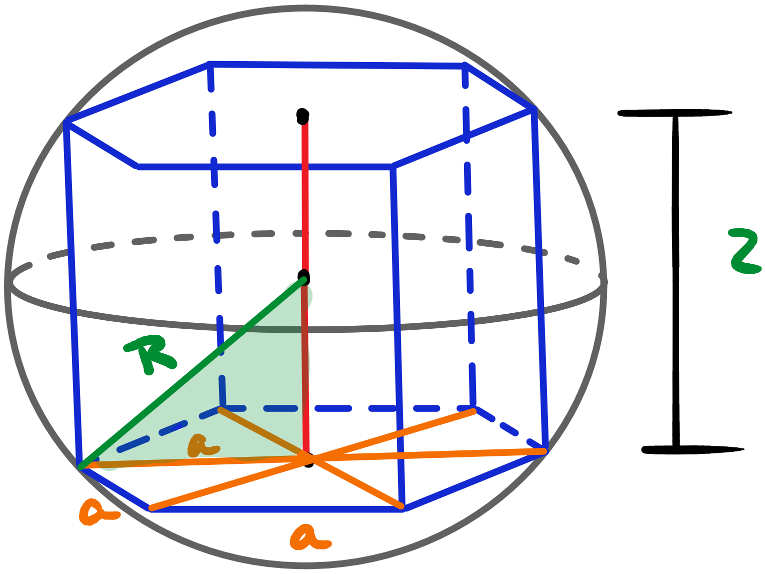
$$V_{cil} = \pi r^2 h$$

$$= \pi \cdot 3^2 \cdot 8$$

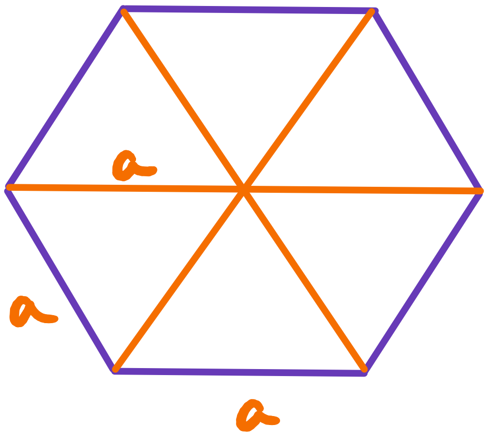
$$\boxed{V_{cil} = 72\pi}$$

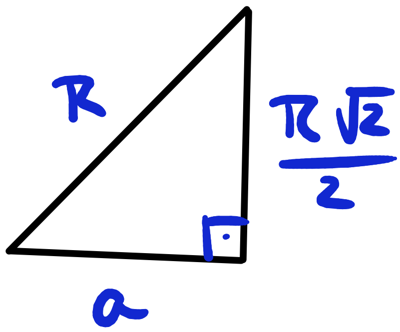
EXEMPLO

NUMA ESFERA DE RAIO R INSCREVE-SE UM PRISMA REGULAR DE BASE HEXAGONAL. CALCULE O VOLUME DO PRISMA SABENDO QUE A DISTÂNCIA DO CENTRO DA ESFERA AO PLANO DA BASE DO PRISMA É $\frac{R\sqrt{2}}{2}$.



$$2 \cdot \frac{R\sqrt{2}}{2} = R\sqrt{2}$$





$$\rightarrow \quad a^2 = R^2 - \left(\frac{R\sqrt{2}}{2}\right)^2$$

$$a^2 = R^2 - \frac{R^2}{2} = \frac{R^2}{2}$$

$$a = \frac{R\sqrt{2}}{2}$$

$$V = A_b \cdot h$$

$$A_b = 6 \cdot \frac{a^2\sqrt{3}}{4}$$

$$= \frac{3}{2} \cdot \frac{R^2}{2} \cdot \sqrt{3}$$

$$A_b = \frac{3R^2\sqrt{3}}{4}$$

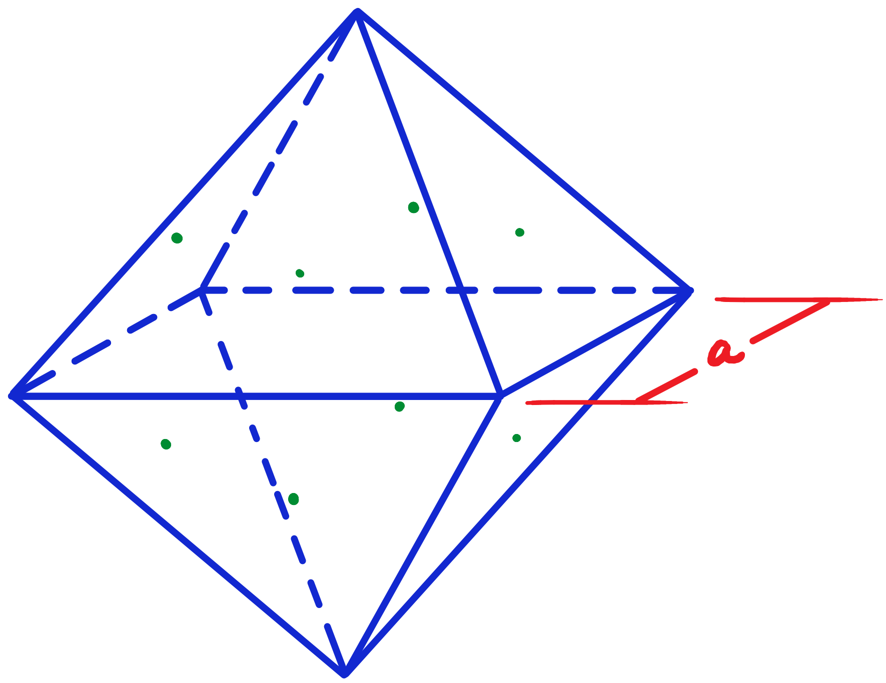
$$V = \frac{3R^2\sqrt{3}}{4} \cdot R\sqrt{2}$$

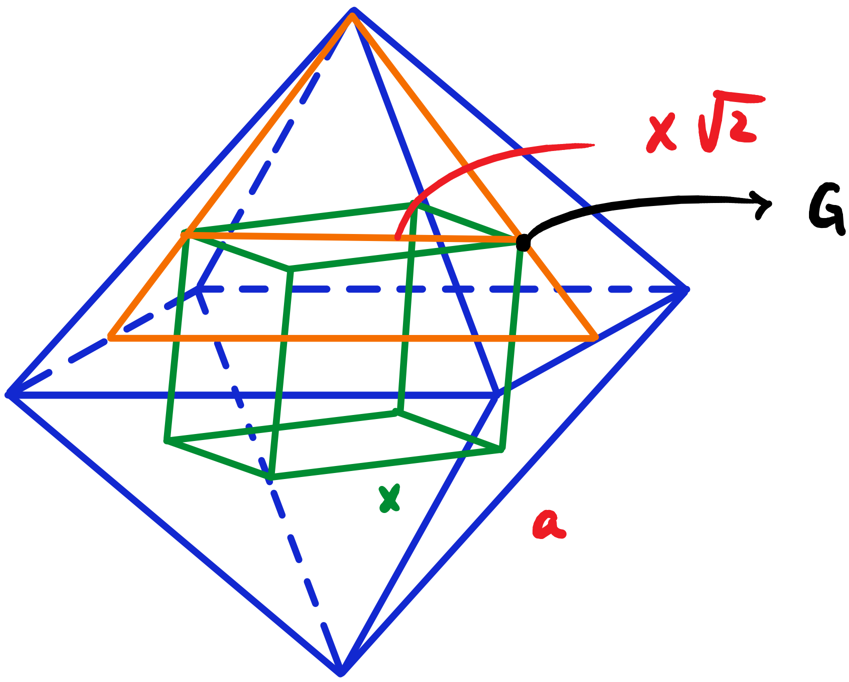
$$\boxed{V = \frac{3R^3\sqrt{6}}{4}}$$

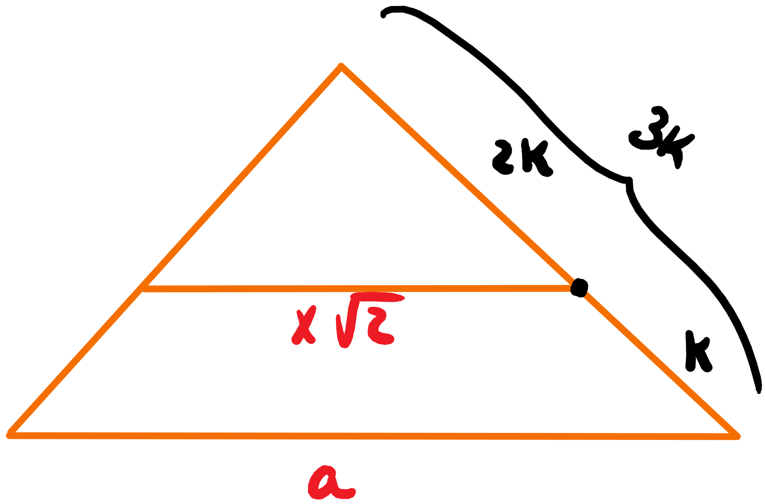
UNIVERSO NARRADO

EXEMPLO

SEJA UM OCTAEDRO REGULAR DE LADO a. CALCULE O VOLUME DO POLIEDRO CUJOS VÉRTICES SÃO OS BARICENTROS DAS FACES DO OCTAEDRO.







$$\frac{x\sqrt{2}}{a} = \frac{2}{3} \rightarrow \quad x = \frac{\cancel{2}a}{3\cancel{\sqrt{2}}} \cdot \frac{\sqrt{2}}{\cancel{\sqrt{2}}}$$

$$\boxed{x = \frac{a\sqrt{2}}{3}}$$

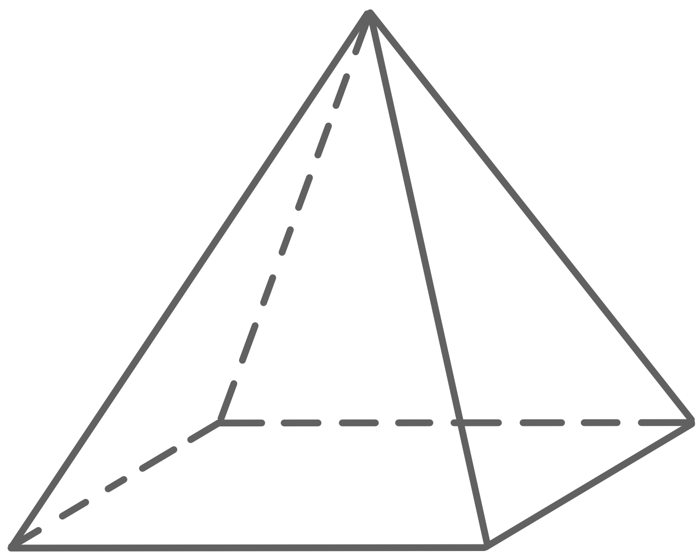
$$V = x^3 = \left(\frac{a\sqrt{2}}{3}\right)^3 = \frac{2a^3\sqrt{2}}{27}$$

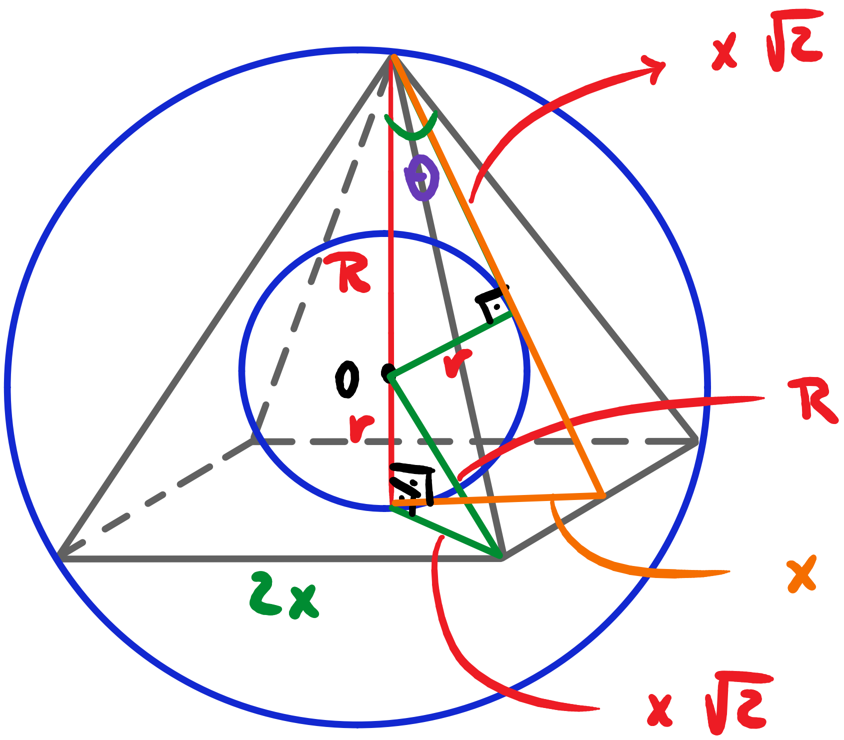
EXEMPLO

SEJA P UMA PIRÂMIDE REGULAR DE BASE QUADRADA.

AS ESFERAS INSCRITA E CIRCUNSCRITA A P POSSUEM CENTROS COINCIDENTES.

DETERMINE A RAZÃO ENTRE AS ÁREAS DESSAS ESFERAS.





$$\frac{A_c}{A_I} = \frac{\cancel{4}\,\cancel{\pi}\,R^2}{\cancel{4}\,\cancel{\pi}\,r^2} = \left(\frac{R}{r}\right)^2$$

$$\frac{r}{x} = \frac{x\sqrt{2}}{R+r}$$

$$x^2\sqrt{2} = (R+r)\,r$$

$$R^2 = r^2 + (x\sqrt{2})^2$$

$$2x^2 = R^2 - r^2 \rightarrow x^2 = \frac{R^2 - r^2}{2}$$

$$\frac{R^2 - r^2}{2} \cdot \sqrt{2} = (R + r)\, r$$

$$\cancel{(R+r)}(R - r)\sqrt{2} = \cancel{(R+r)} \cdot r \cdot 2$$

$$\frac{R - r}{r} = \frac{\sqrt{2} \cdot \cancel{\sqrt{2}}}{\cancel{\sqrt{2}}}$$

$$\frac{R}{r} - 1 = \sqrt{2} \rightarrow \boxed{\frac{R}{r} = \sqrt{2} + 1}$$

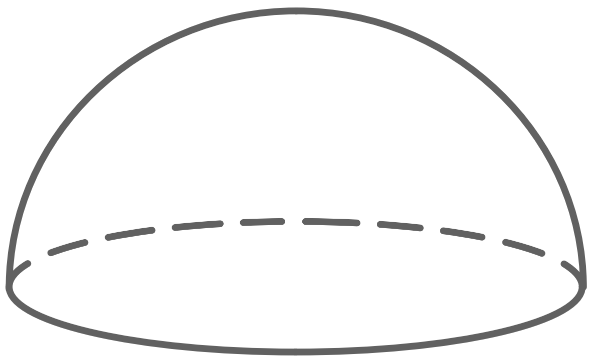
$$\frac{A_C}{A_I} = (\sqrt{2} + 1)^2 = 2 + 1 + 2\sqrt{2} =$$

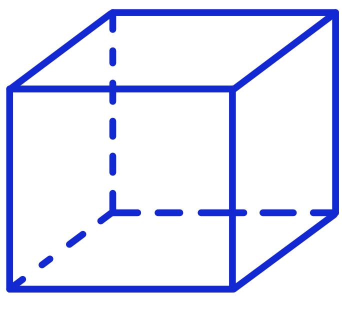
$$\boxed{\frac{A_C}{A_I} = 3 + 2\sqrt{2}}$$

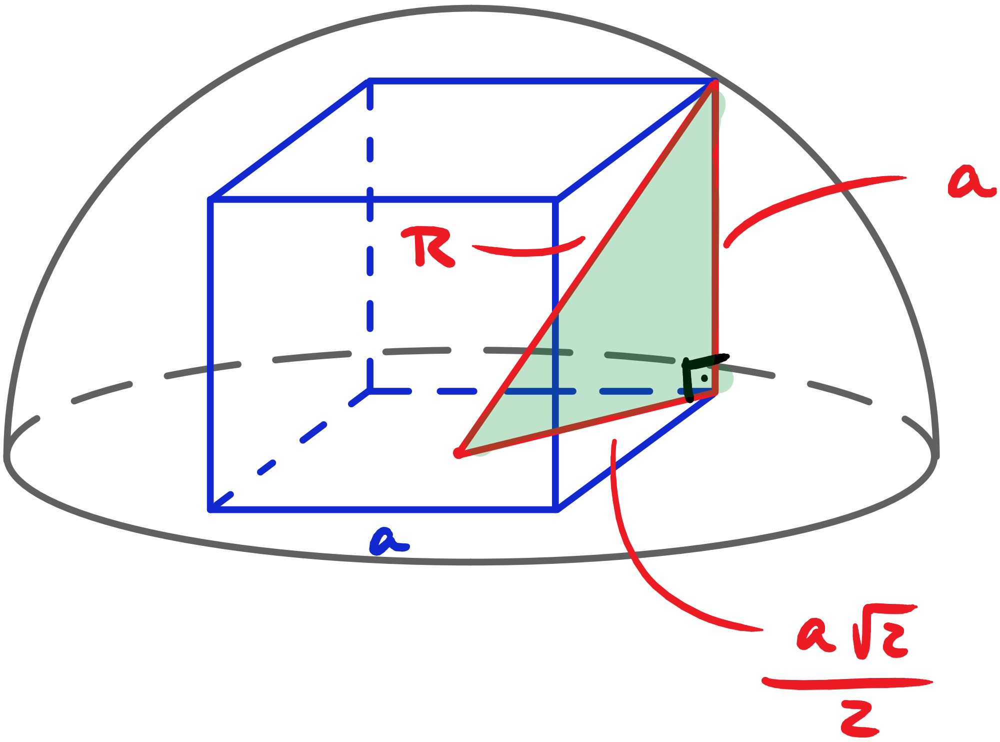
EXEMPLO

UM CUBO DE ARESTA a ESTÁ INSCRITO EM UMA SEMIESFERA DE RAIO R.

CALCULE A RAZÃO : R/a







$$R^2 = a^2 + \left(\frac{a\sqrt{2}}{2}\right)^2$$

$$R^2 = a^2 + \frac{a^2}{2}$$

$$R^2 = \frac{3a^2}{2}$$

$$\left(\frac{R}{a}\right)^2 = \frac{3}{2}$$

$$\frac{R}{a} = \frac{\sqrt{3}}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}}$$

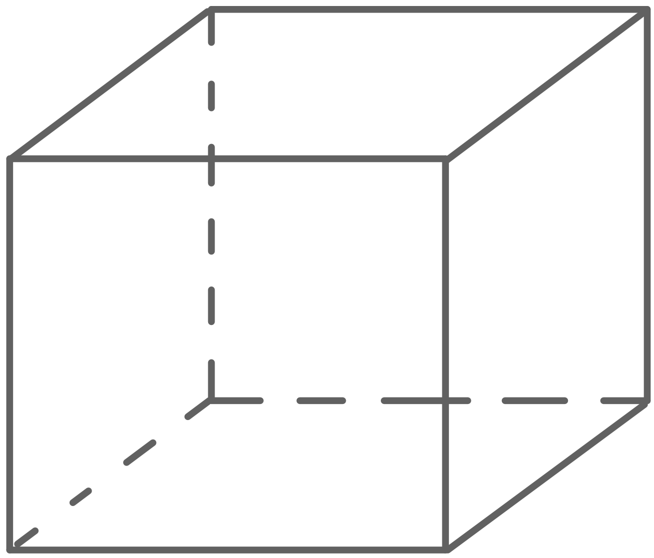
$$\boxed{\frac{R}{a} = \frac{\sqrt{6}}{2}}$$

UNIVERSO NARRADO

EXEMPLO

SEJA UM CUBO DE ARESTA a.

CALCULE O VOLUME DO TETRAEDRO REGULAR CUJOS VÉRTICES SÃO 4 DOS VÉRTICES DESSE CUBO.

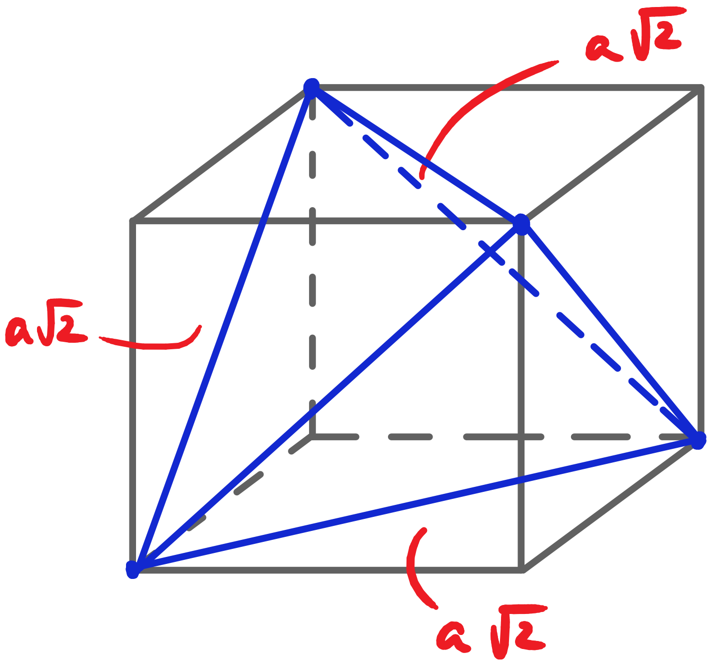


EXEMPLO

SEJA UM CUBO DE ARESTA a.

CALCULE O VOLUME DO TETRAEDRO REGULAR CUJOS VÉRTICES SÃO 4 DOS VÉRTICES DESSE CUBO.

$$V_T = \frac{x^3\sqrt{2}}{12}$$

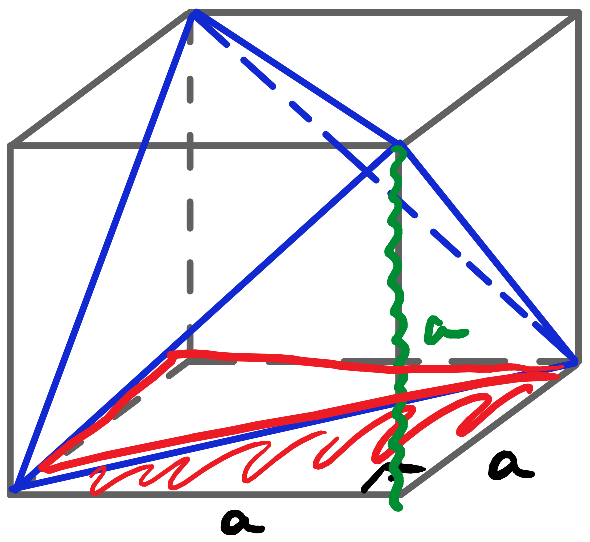


ARESTA $a\sqrt{2}$

$$V_T = \frac{x^3\sqrt{2}}{12} = \frac{(a\sqrt{2})^3 \cdot \sqrt{2}}{12}$$

$$= \frac{a^3 \cdot \cancel{2} \cdot \cancel{2}}{\cancel{12}\ 3}$$

$$= \frac{a^3}{3}$$



$$V_P = \frac{1}{3} \cdot \underbrace{\frac{a^2}{2}}_{A_b} \cdot \underbrace{a}_{h}$$

$$V_P = \frac{a^3}{6}$$

$$V_T = V_{CUBO} - 4 . V_P$$

$$= a^3 - 4 \cdot \frac{a^3}{6}$$

$$= a^3 - \frac{2a^3}{3}$$

$$\boxed{V_T = \frac{a^3}{3}}$$