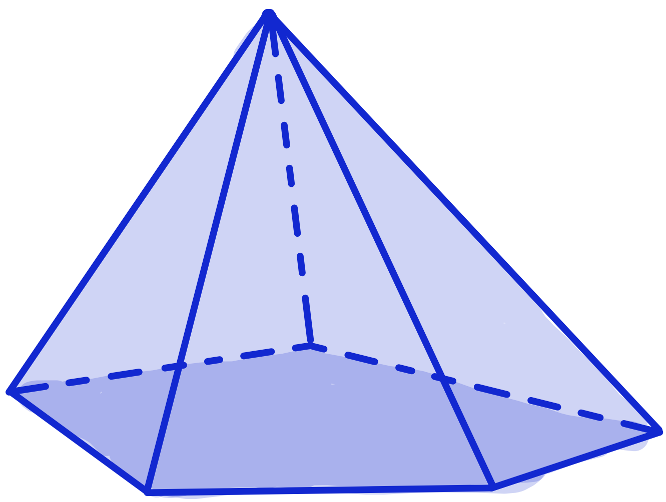
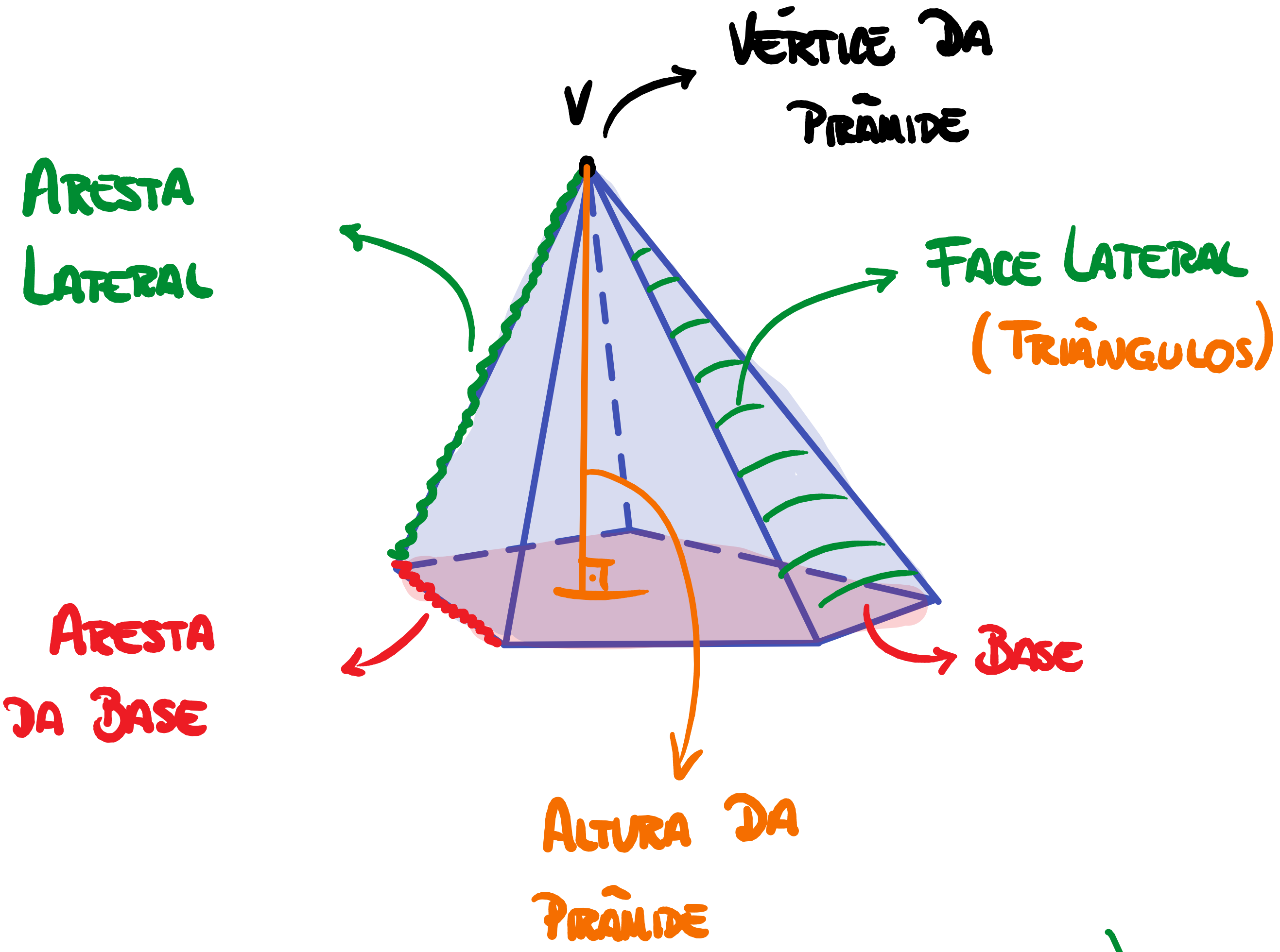
# PIRÂMIDES

## DEFINIÇÃO

POLIEDRO CONVEXO OBTIDO AO UNIR OS VÉRTICES DE UM POLÍGONO A UM VÉRTICE FORA DO PLANO DESSE POLÍGONO.

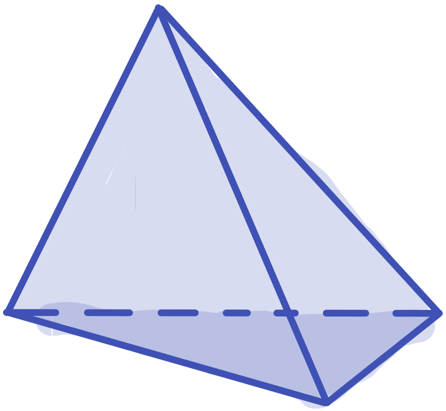


# ELEMENTOS DA PIRÂMIDE

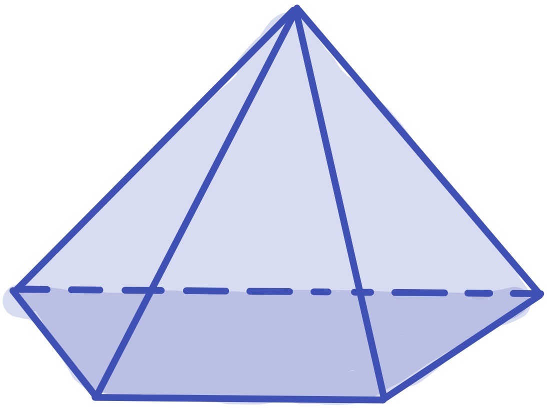



(DISTÂNCIA DO VÉRTICE AO PLANO DA BASE)

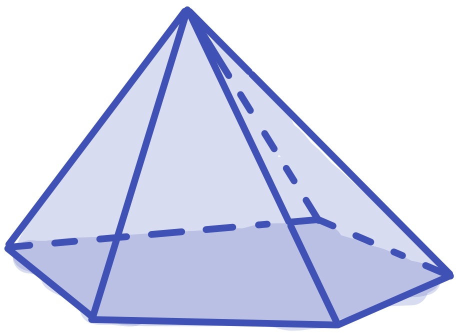
# CLASSIFICAÇÃO



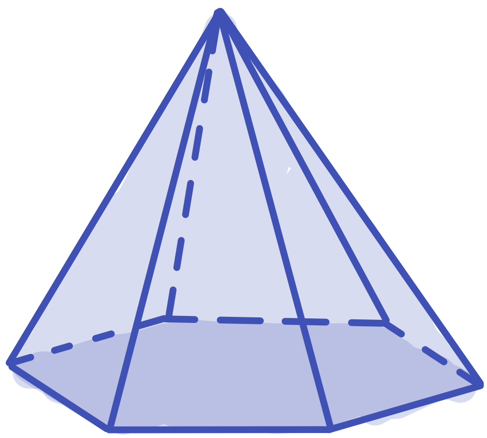
PIRÂMIDE
TRIANGULAR



PIRÂMIDE
QUADRILATERAL



PIRÂMIDE
PENTAGONAL

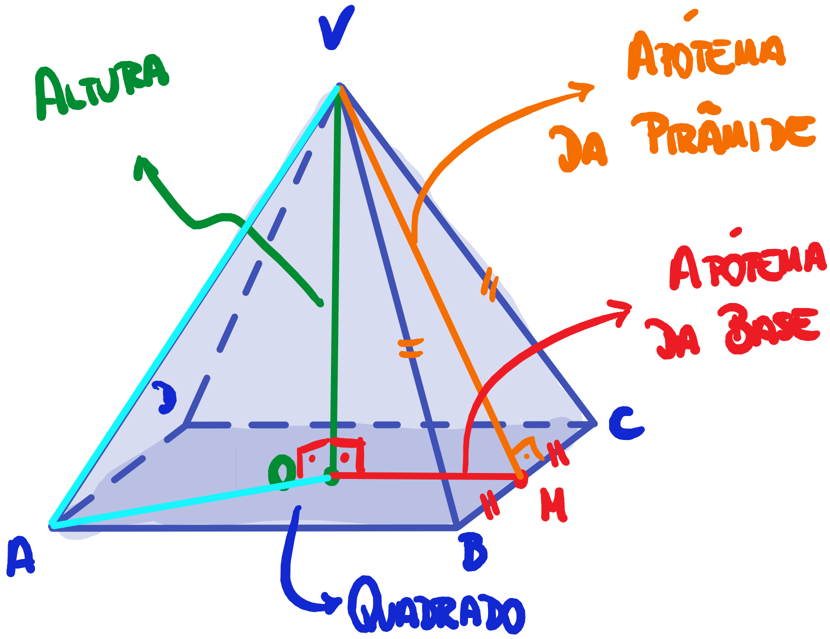


PIRÂMIDE
HEXAGONAL

# PIRÂMIDE REGULAR

- BASE: POLÍGONO REGULAR
- ARESTAS LATERAIS SÃO CONGRUENTES.

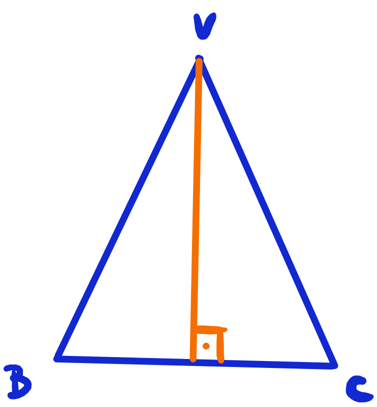
CONSEQUÊNCIA: PROJEÇÃO DO VÉRTICE SOBRE O CENTRO DA BASE.




$$\boxed{VM^2 = OM^2 + OV^2}$$

$$\boxed{VA^2 = OA^2 + OV^2}$$

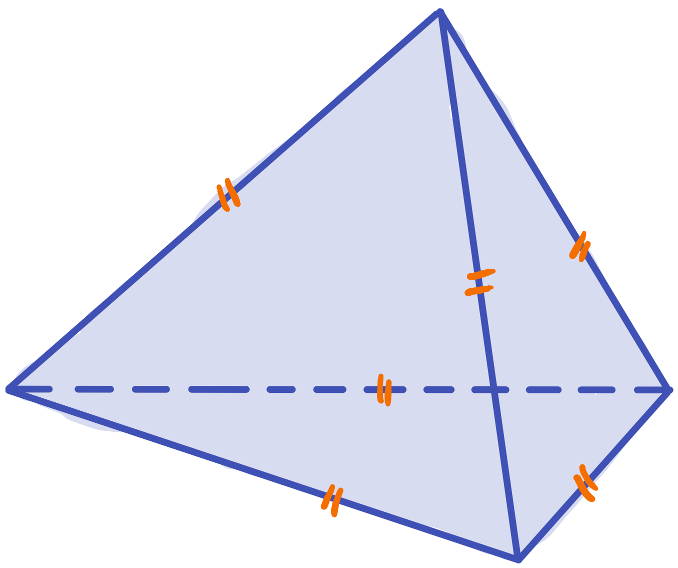
$$\boxed{VC^2 = MC^2 + MV^2}$$



# TETRAEDRO REGULAR

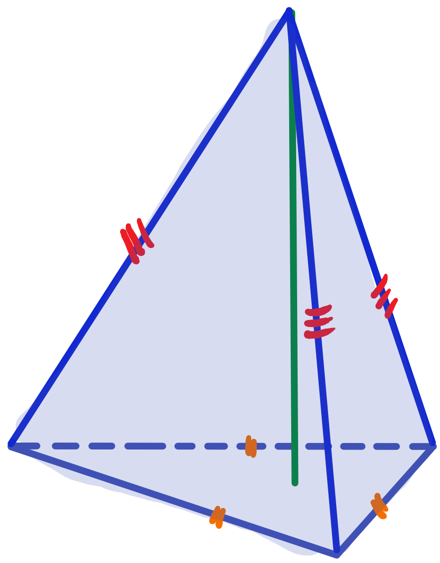
PIRÂMIDE TRIANGULAR REGULAR COM TODAS AS ARESTAS EQUIVALENTES.

CONSEQUÊNCIA: TODAS AS FACES SÃO TRIÂNGULOS EQUILÁTEROS



PIRÂMIDE REGULAR ✓

TETRAEDRO REGULAR ✓



PIRAMIDE REGULAR ✓

~~TETRAEDRO REGULAR~~

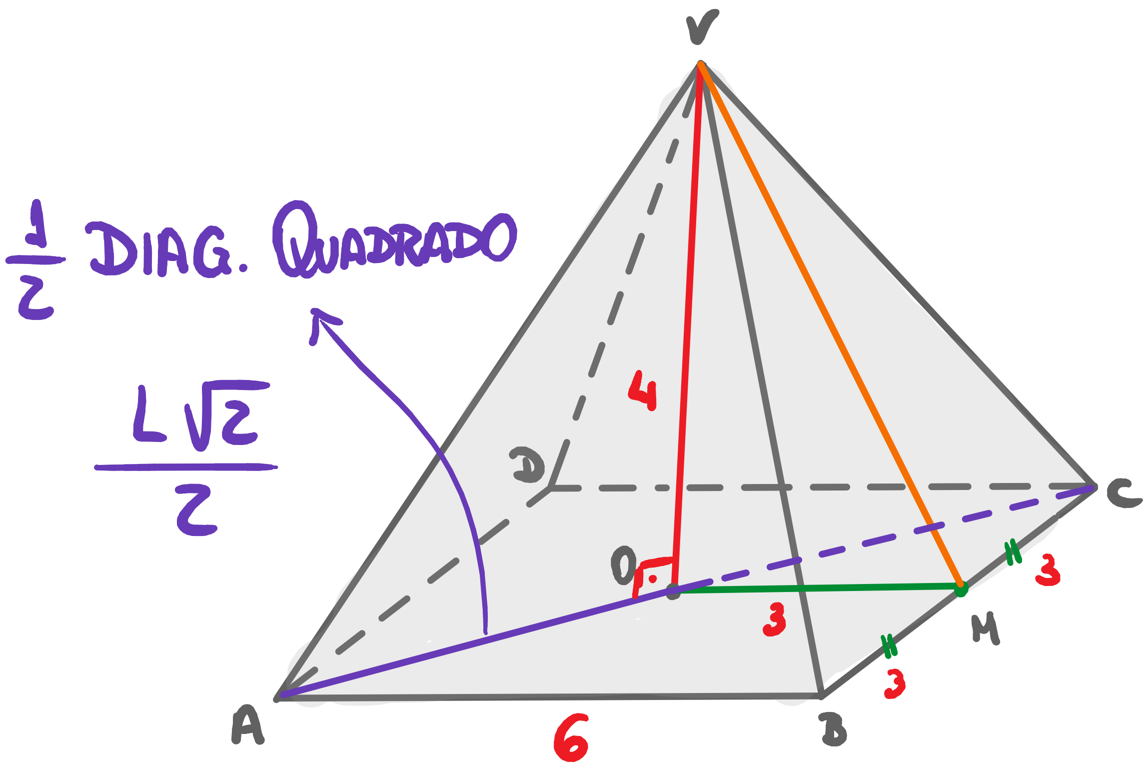
# EXEMPLO

SEJA UMA PIRÂMIDE QUADRILATERAL REGULAR DE ALTURA 4 E ARESTA DA BASE 6.

CALCULE: a. APÓTEMA DA BASE.

b. APÓTEMA DA PIRÂMIDE.

c. ARESTA LATERAL.

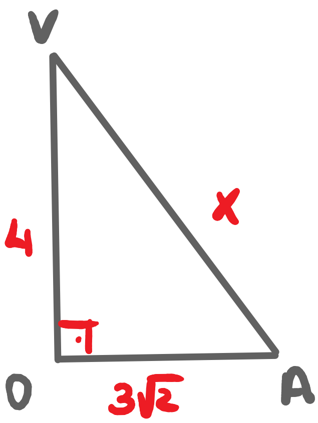


a. $OM = 3$

b. $VM^2 = OM^2 + OV^2$

$$\boxed{VM = 5}$$

c) $\Delta VOA$



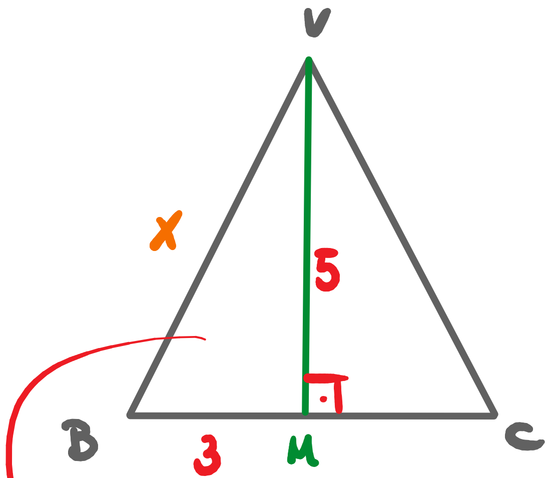

$$x^2 = 4^2 + (3\sqrt{2})^2$$

$$x^2 = 16 + 18$$

$$x^2 = 34$$

$$\boxed{x = \sqrt{34}}$$

c) $\Delta VBC$




$$x^2 = 3^2 + 5^2$$

$$x^2 = 9 + 25$$

$$x^2 = 34$$

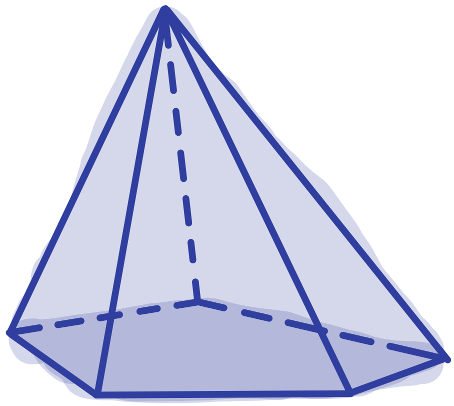
$$\boxed{x = \sqrt{34}}$$

# Áreas

$A_b$ : Área da Base

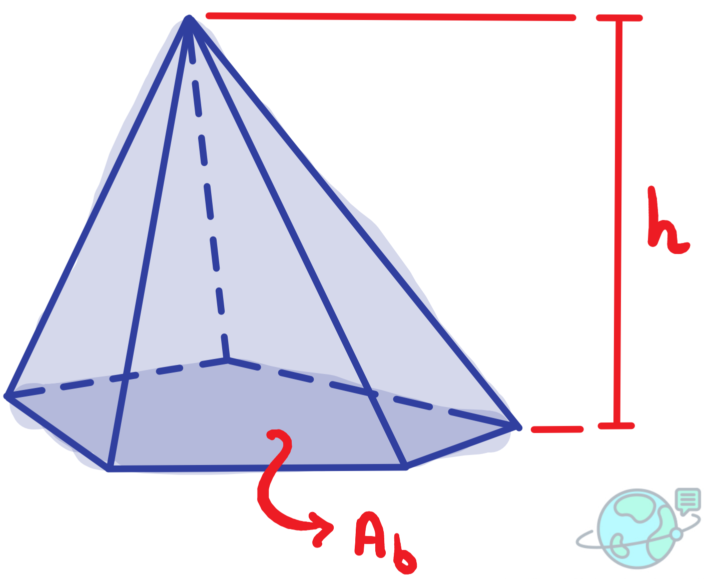
$A_L$ : Área Lateral

$A_T$ : Área Total



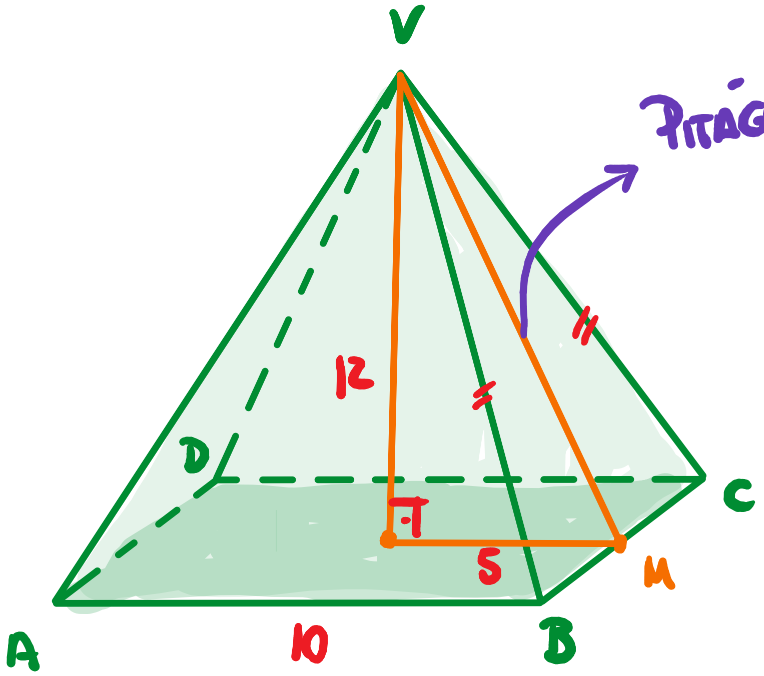
$$\boxed{A_T = A_b + A_L}$$

# Volume




$$\boxed{V = \frac{1}{3} \cdot A_b \cdot h}$$

# PIRÂMIDE REGULAR



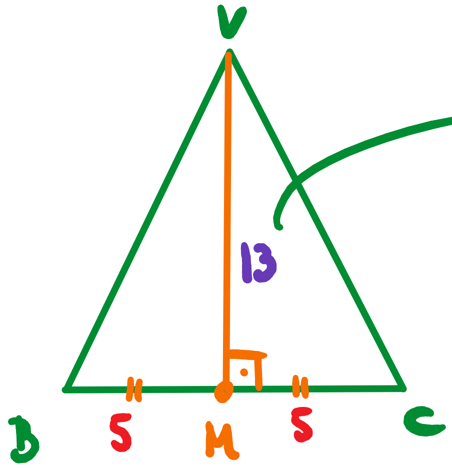

$VM = 13$

$AB = 10$

ALTURA = 12

• ÁREA DA BASE: $A_b = 10^2 \rightarrow A_b = 100$

- **ÁREA LATERAL**




$A_{VBC} = \frac{1}{2} \cdot 10 \cdot 13$

$A_{VBC} = 65$

$A_L = 4 \cdot 65 \rightarrow \boxed{A_L = 260}$

**ÁREA TOTAL**: $A_T = A_b + A_L$

$\boxed{A_T = 360}$

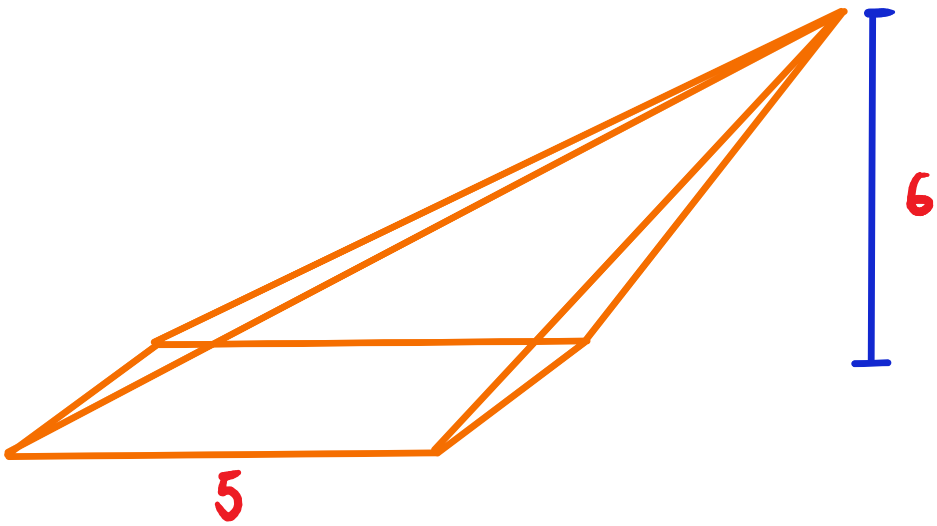
- **VOLUME**: $V = \frac{1}{3} \cdot A_b \cdot h$

$V = \frac{1}{3} \cdot 100 \cdot 12$

$\boxed{V = 400}$

## EXEMPLO

CALCULE O VOLUME DE UMA PIRÂMIDE CUJA BASE É UM QUARADO DE LADO 5 E ALTURA 6.




$$V = \frac{1}{3} \cdot A_b \cdot h$$

$$V = \frac{1}{3} \cdot 5^2 \cdot 6$$

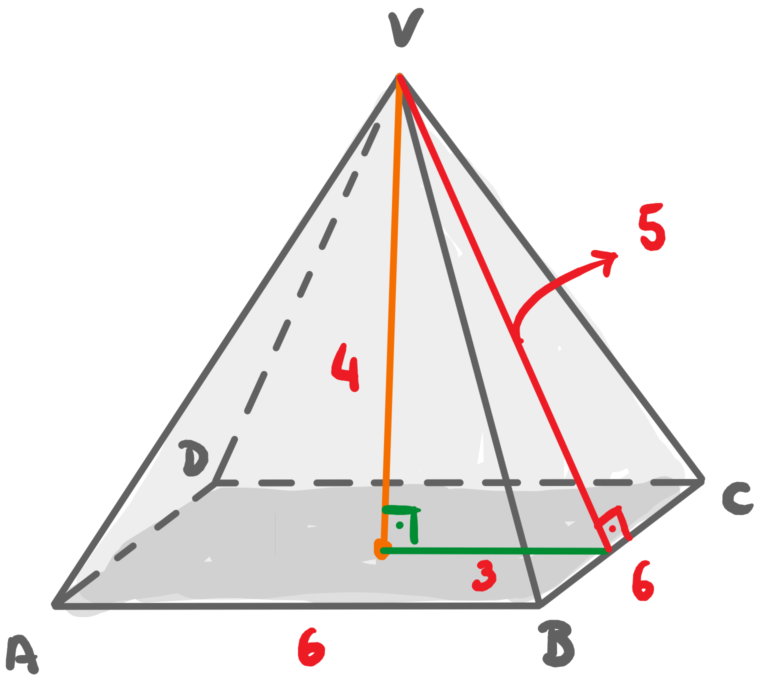
$$\boxed{V = 50}$$

# EXEMPLO

SEJA UMA PIRÂMIDE QUADRILATERAL REGULAR DE ALTURA 4 E ARESTA DA BASE 6.

CALCULE: a. A ÁREA TOTAL DA PIRÂMIDE

b. O VOLUME DA PIRÂMIDE




$A_b = 6^2 \rightarrow \boxed{A_b = 36}$

$A_L = 4 \cdot \frac{1}{2} \cdot 6 \cdot 5 \rightarrow \boxed{A_L = 60}$

$A_T = 36 + 60 \rightarrow \boxed{A_T = 96}$

$V = \frac{1}{3} \cdot A_b \cdot h$

$V = \frac{1}{3} \cdot 36 \cdot 4$

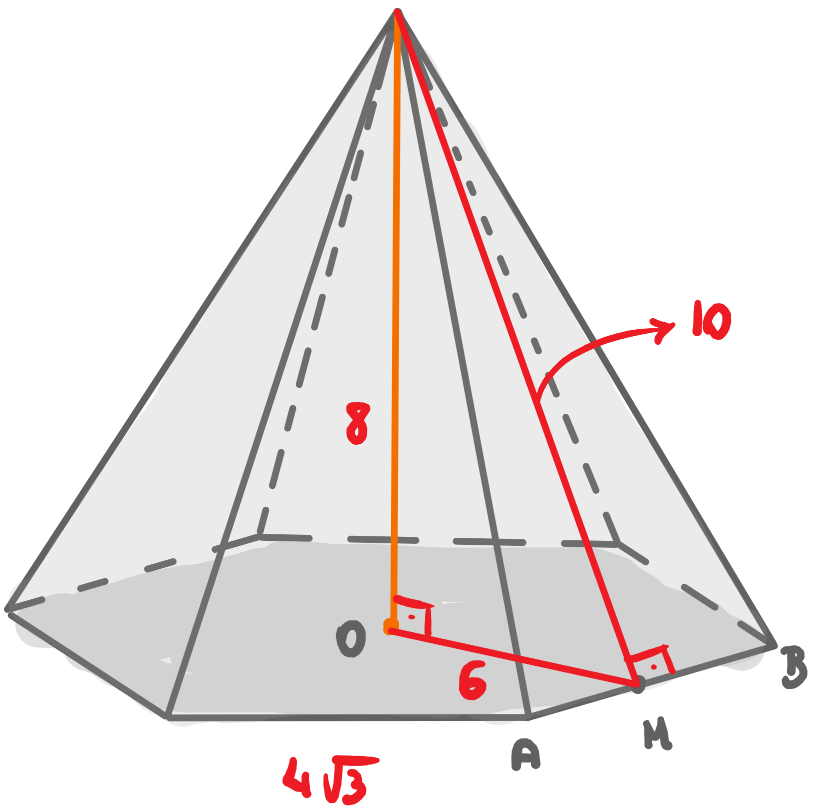
$\boxed{V = 48}$

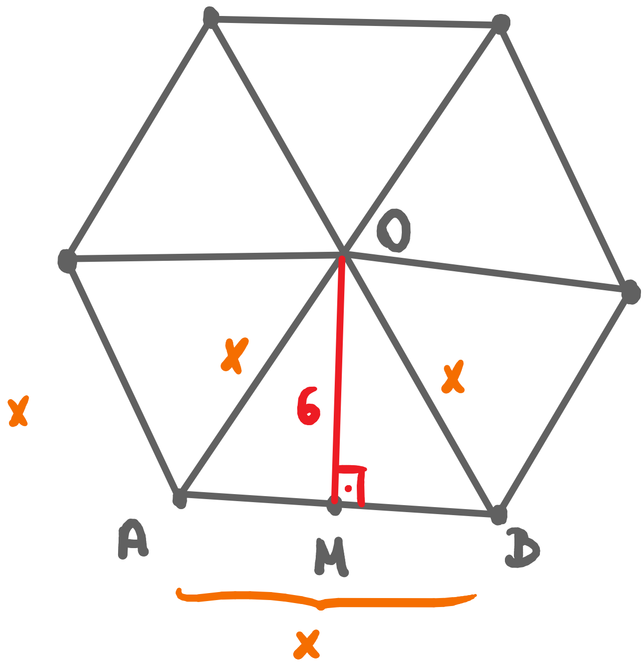
# EXEMPLO

SEJA UMA PIRÂMIDE HEXAGONAL REGULAR DE ALTURA 8 E APÓTEMA 10.

CALCULE: a. A ÁREA TOTAL DA PIRÂMIDE

b. O VOLUME DA PIRÂMIDE






$$h_{\Delta EQ} = \frac{\ell\sqrt{3}}{2}$$

$$6 = \frac{x\sqrt{3}}{2}$$

$$x = \frac{12}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}}$$

$$\boxed{x = 4\sqrt{3}}$$

$$A_b = 6 \cdot A_{\Delta EQ} = 6 \cdot \frac{(4\sqrt{3})^2\sqrt{3}}{4} = \frac{6 \cdot \overset{4}{\cancel{16}} \cdot 3 \cdot \sqrt{3}}{\cancel{4}}$$

$$\boxed{A_b = 72\sqrt{3}}$$

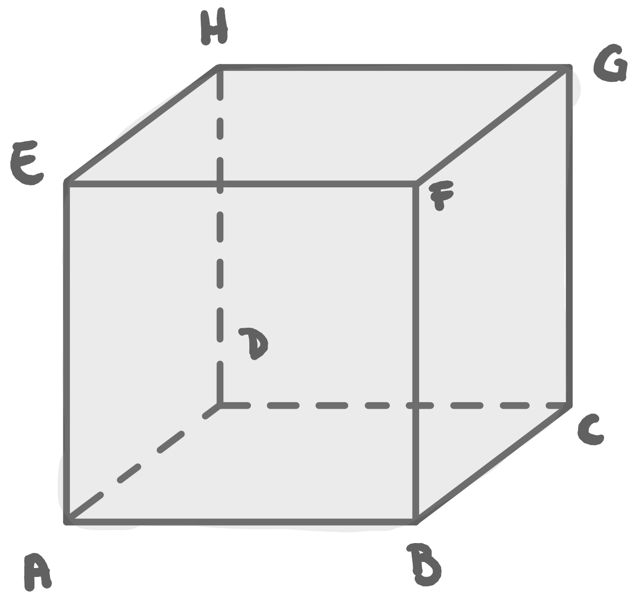
$$A_L = 6 \cdot \frac{1}{2} \cdot b \cdot h = 3 \cdot 4\sqrt{3} \cdot 10 \rightarrow \boxed{A_L = 120\sqrt{3}}$$

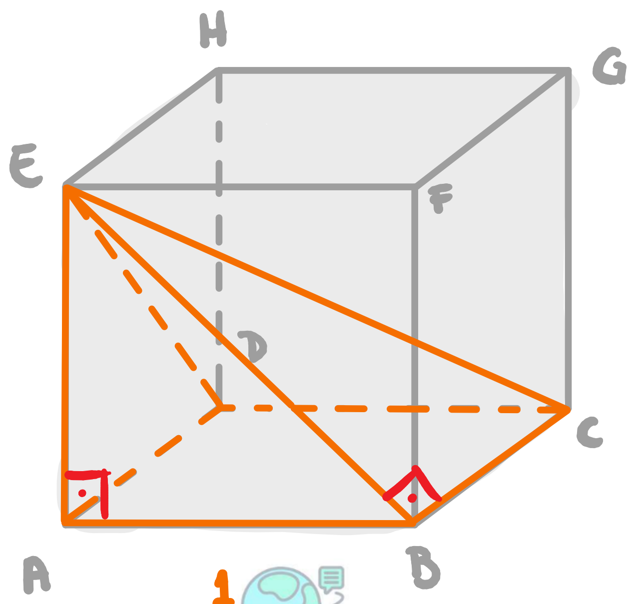
$$A_T = A_b + A_L \rightarrow \boxed{A_T = 192\sqrt{3}}$$

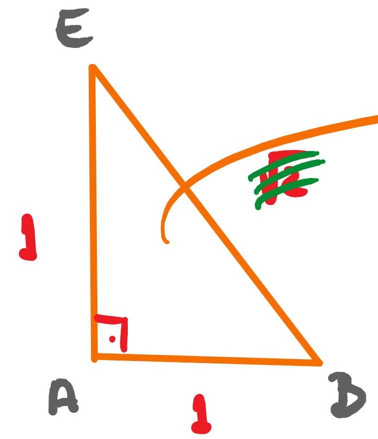
$$V = \frac{1}{3} \cdot A_b \cdot h = \frac{1}{\cancel{3}} \cdot \overset{24}{\cancel{72}}\sqrt{3} \cdot 8 \rightarrow \boxed{V = 192\sqrt{3}}$$

## EXEMPLO

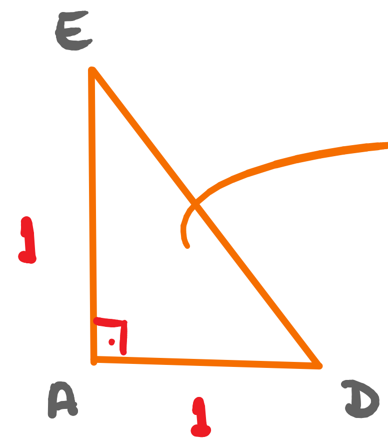
CALCULE A ÁREA LATERAL DA PIRÂMIDE ABCDE, SABENDO QUE ABCDEFGH É UM CUBO DE ARESTA 1.



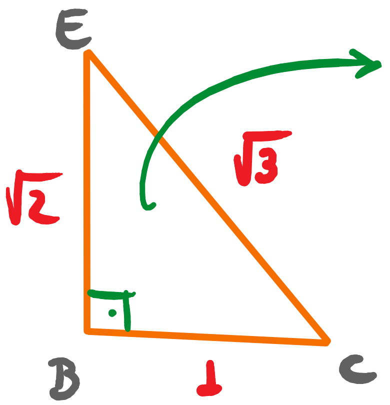






$$A_1 = \frac{1}{2} \cdot 1 \cdot 1 \rightarrow \boxed{A_1 = \frac{1}{2}}$$



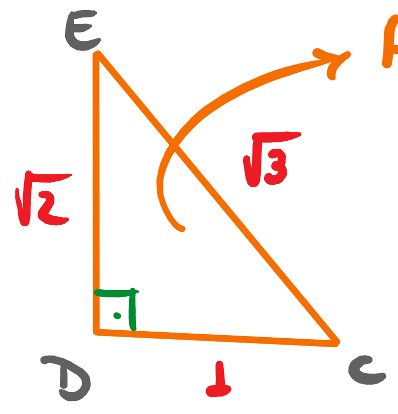

$$A_2 = A_1 \rightarrow \boxed{A_2 = \frac{1}{2}}$$




$$\text{Pit. } (\sqrt{3})^2 = (\sqrt{2})^2 + 1^2$$

$$3 \overset{\checkmark}{=} 3 \quad (\Delta \text{ É RETÂNGULO})$$

$$A_3 = \frac{1}{2} \cdot 1 \cdot \sqrt{2} \rightarrow \boxed{A_3 = \frac{\sqrt{2}}{2}}$$




$$A_4 = A_3$$

$$A_4 = \frac{\sqrt{2}}{2}$$

$$A_L = A_1 + A_2 + A_3 + A_4$$

$$A_L = 2 \cdot \frac{1}{2} + 2 \cdot \frac{\sqrt{2}}{2}$$

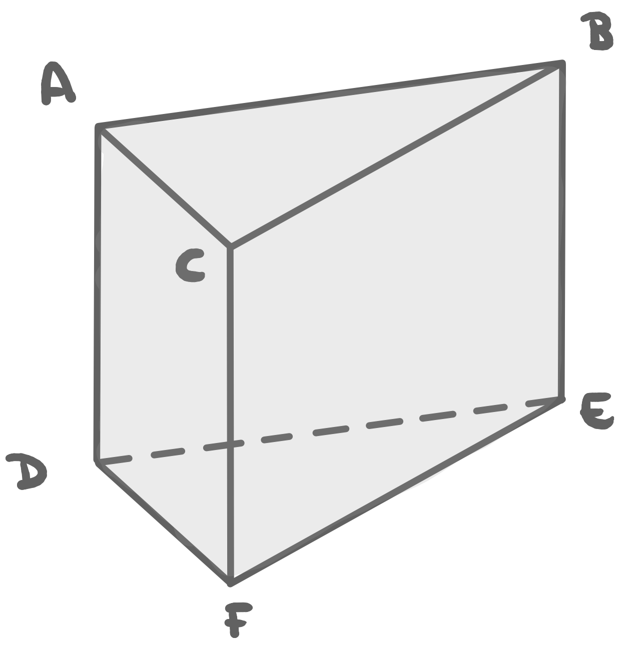
$$\boxed{A_L = 1 + \sqrt{2}}$$

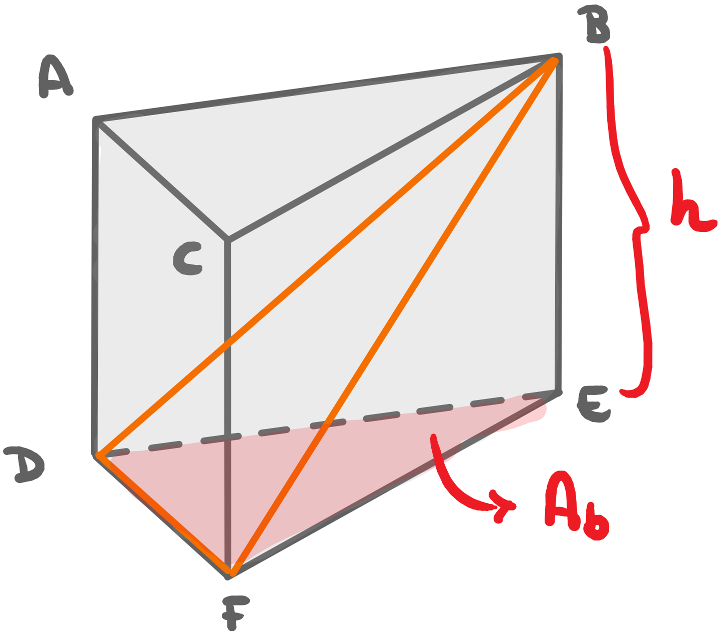
# EXEMPLO

O PRISMA TRIANGULAR RETO É DIVIDIDO EM DOIS SÓLIDOS PELO PLANO BDF.

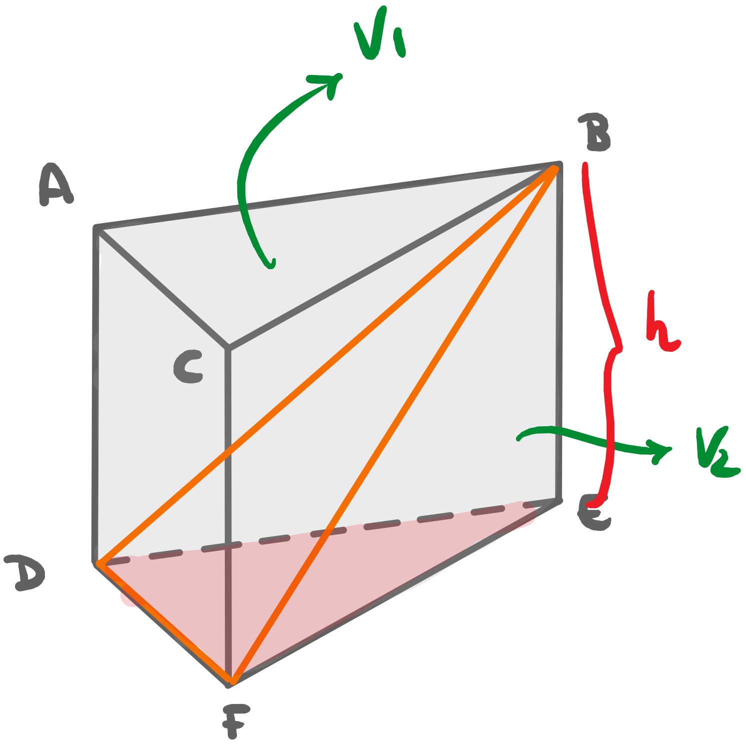
SÓLIDO 1: ACFDB, DE VOLUME V1.
SÓLIDO 2: DEFB, DE VOLUME V2

CALCULE A RAZÃO V1/ V2.






$$V_{TOTAL} = A_b \cdot h$$



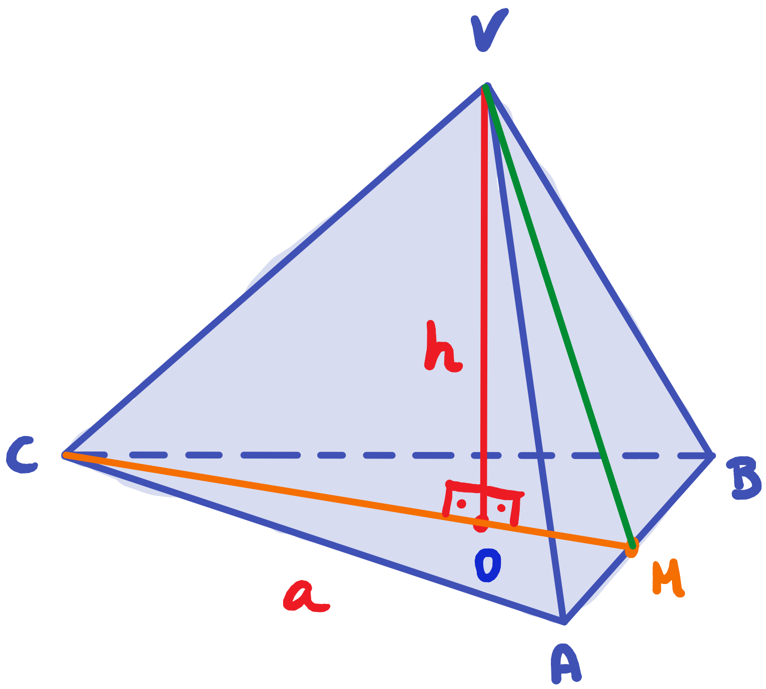

$$V_2 = \frac{1}{3} \cdot A_b \cdot h$$

$$\boxed{V_2 = \frac{1}{3} V_{TOTAL}}$$

$$\boxed{V_1 = \frac{2}{3} V_{TOTAL}}$$

$$\boxed{\frac{V_1}{V_2} = 2}$$

# Tetraedro Regular



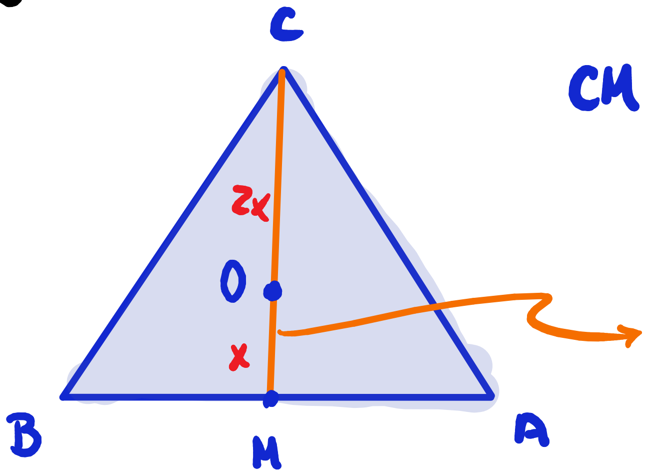

$$\boxed{A_b = \frac{a^2\sqrt{3}}{4}}$$

$$\boxed{A_T = 4 \cdot A_b = a^2\sqrt{3}}$$

$$\boxed{h = \frac{a\sqrt{6}}{3}}$$

$$\boxed{V = \frac{a^3\sqrt{2}}{12}}$$

**BASE**



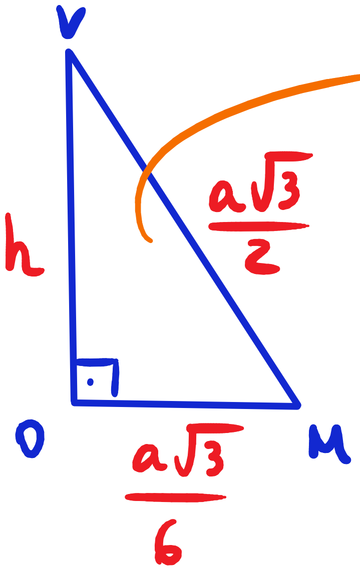

$$CM = h_{EQ} \rightarrow CM = \frac{a\sqrt{3}}{2}$$

$$OM = \frac{1}{3} CM = \frac{1}{3} \cdot \frac{a\sqrt{3}}{2}$$

$$\boxed{OM = \frac{a\sqrt{3}}{6}} \quad \boxed{OC = \frac{a\sqrt{3}}{3}}$$

UNIVERSO NARRADO

$\Delta VOM$




T. Pitágoras:

$$h^2 = \left(\frac{a\sqrt{3}}{2}\right)^2 - \left(\frac{a\sqrt{3}}{6}\right)^2$$

$$h^2 = \frac{a^2 . 3}{4} - \frac{a^2 . 3}{36}$$

$$h^2 = \frac{3}{4}a^2 - \frac{1}{12}a^2$$

$$h^2 = \frac{\cancel{8}^{\,2}}{\cancel{12}_{\,3}}a^2$$

$$h = a\,\frac{\sqrt{2}}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}}$$

$$\boxed{h = \frac{a\sqrt{6}}{3}}$$

$$V = \frac{1}{3} \cdot \frac{a^2\sqrt{\cancel{3}}}{4} \cdot \frac{a\sqrt{\cancel{3}} . \sqrt{2}}{\cancel{3}} \rightarrow \boxed{V = \frac{a^3\sqrt{2}}{12}}$$

## EXEMPLO

CONSIDERE UM TETRAEDRO REGULAR DE ARESTA 2.

CALCULE: a. A ÁREA TOTAL DO TETRAEDRO

b. O VOLUME DO TETRAEDRO

$$A_b = \frac{a^2\sqrt{3}}{4} \quad ; \quad A_T = 4 . A_b = a^2\sqrt{3}$$

$$h = \frac{a\sqrt{6}}{3} \quad ; \quad V = \frac{a^3\sqrt{2}}{12}$$

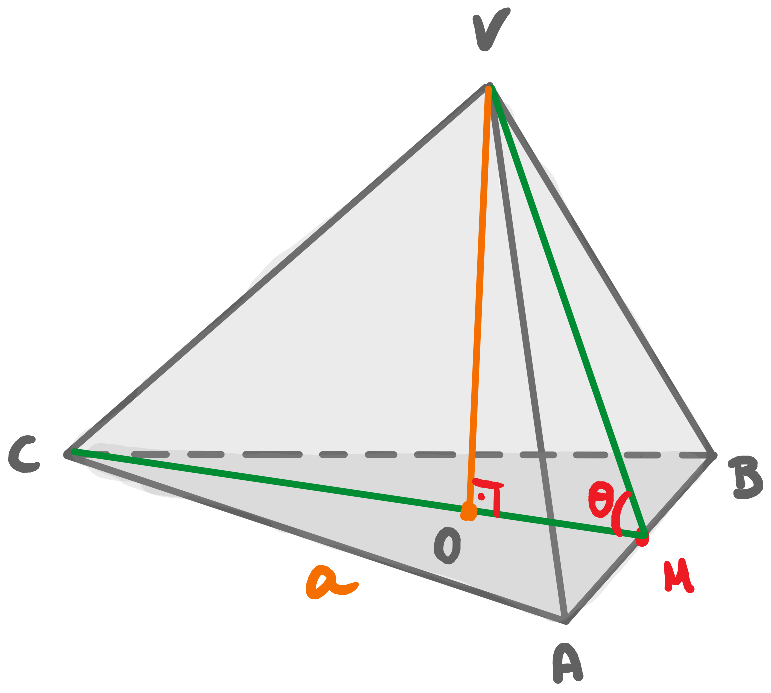
$$\underline{a = 2}$$

a) $A_T = 2^2\sqrt{3} = 4\sqrt{3}$

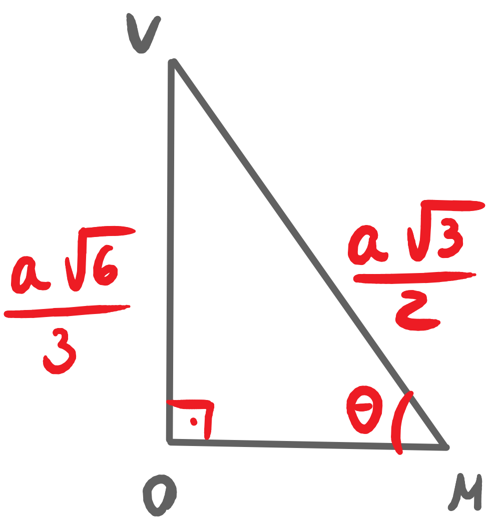
b) $V = \dfrac{2^3\sqrt{2}}{12} = \dfrac{\cancel{8}^{\,2}\sqrt{2}}{\cancel{12}_{\,3}} = \dfrac{2\sqrt{2}}{3}$

# EXEMPLO

CALCULE O SENO DO ÂNGULO FORMADO ENTRE DUAS FACES DE UM TETRAEDRO REGULAR.




$$\operatorname{sen}\theta = ?$$




$$\operatorname{sen}\theta = \frac{\frac{a\sqrt{6}}{3}}{\frac{a\sqrt{3}}{2}}$$

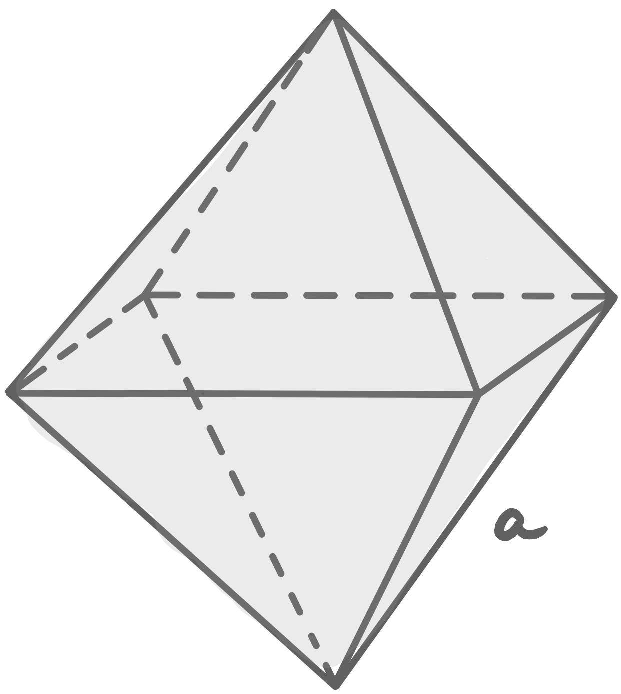
$$\sqrt{6} \rightarrow \cancel{\sqrt{3}} \cdot \sqrt{2}$$

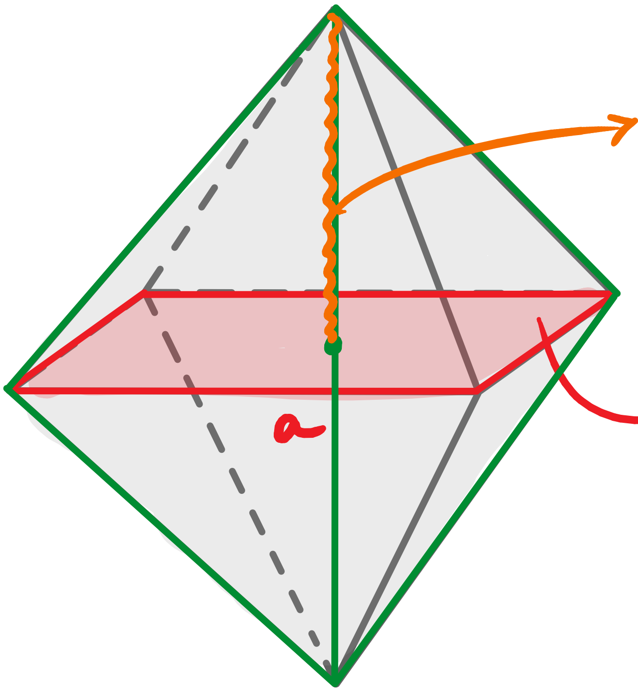
$$\operatorname{sen}\theta = \frac{\cancel{a}\sqrt{6}}{3} \cdot \frac{2}{\cancel{a}\cancel{\sqrt{3}}}$$

$$\boxed{\operatorname{sen}\theta = \frac{2\sqrt{2}}{3}}$$

# EXEMPLO

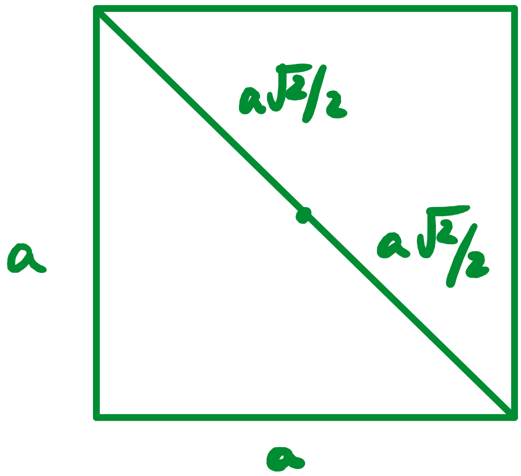
CALCULE O VOLUME DE UM OCTAEDRO REGULAR DE ARESTA $a$.






$\frac{1}{2} \cdot a\sqrt{2}$ : $h = \frac{a\sqrt{2}}{2}$

$A_b = a^2$



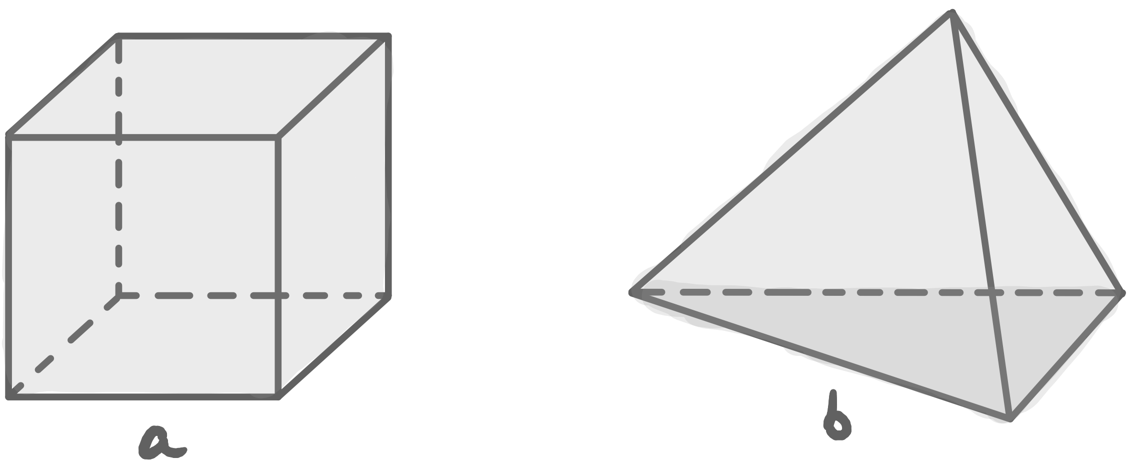

$$V_{OCT} = 2 \cdot V_{PIR}$$

$$= \cancel{2} \cdot \frac{1}{3} \cdot a^2 \cdot \frac{a\sqrt{2}}{\cancel{2}}$$

$$\boxed{V_{OCT} = \frac{a^3\sqrt{2}}{3}}$$

## EXEMPLO

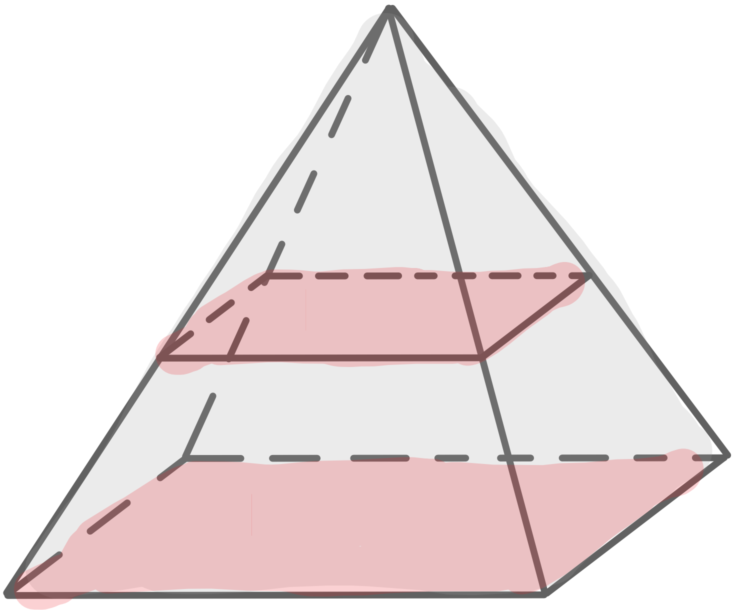
CALCULE A RAZÃO ENTRE A ARESTA a DE UM CUBO E A ARESTA b DE UM TETRAEDRO REGULAR SABENDO QUE ELES TEM MESMO VOLUME.

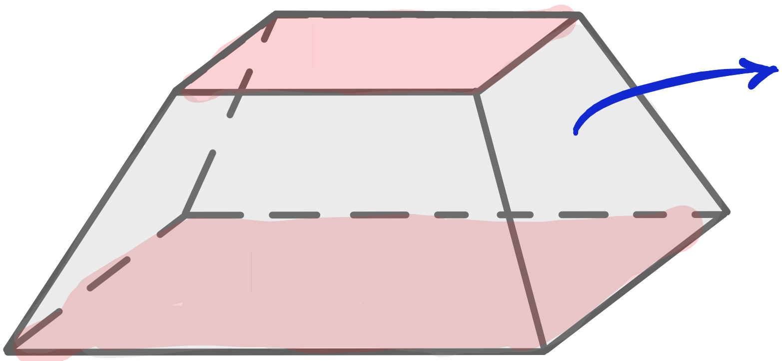



$$V_1 = a^3 \qquad V_2 = \frac{b^3\sqrt{2}}{12}$$

$$a^3 = \frac{b^3\sqrt{2}}{12} \rightarrow \left(\frac{a}{b}\right)^3 = \frac{\sqrt{2}}{12} \therefore \boxed{\frac{a}{b} = \sqrt[3]{\frac{\sqrt{2}}{12}}}$$

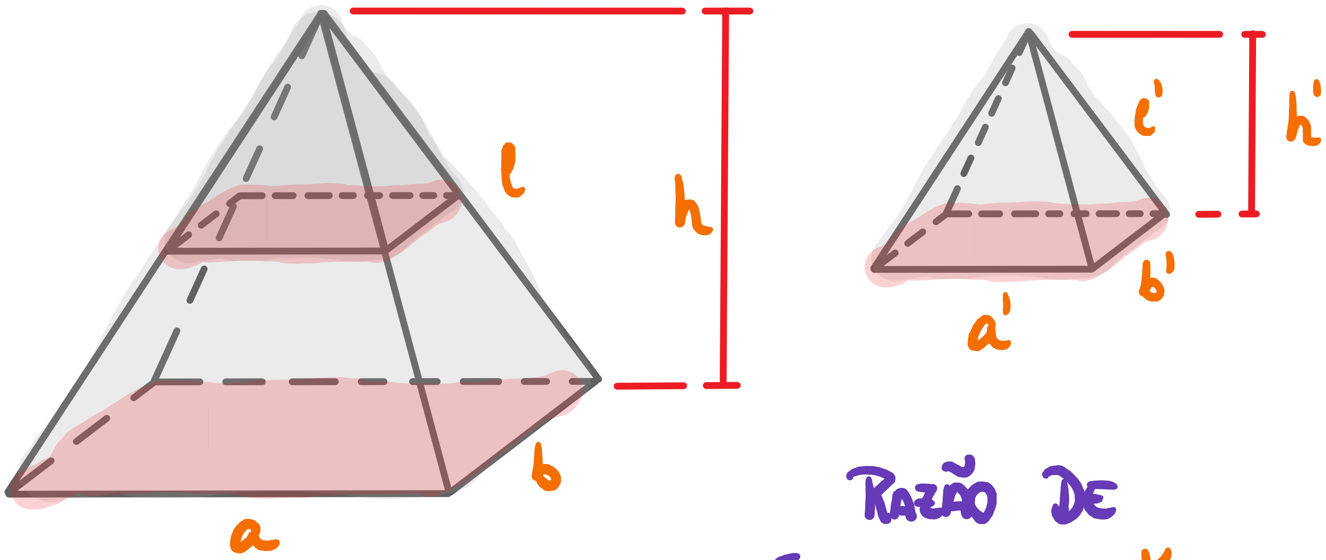
# TRONCO DE PIRÂMIDE





TRONCO DE PIRÂMIDE

# PIRÂMIDES SEMELHANTES




RAZÃO DE SEMELHANÇA : K

COMPRIMENTO :

$$\boxed{\frac{a'}{a} = \frac{b'}{b} = \frac{h'}{h} = \frac{l'}{l} = K}$$

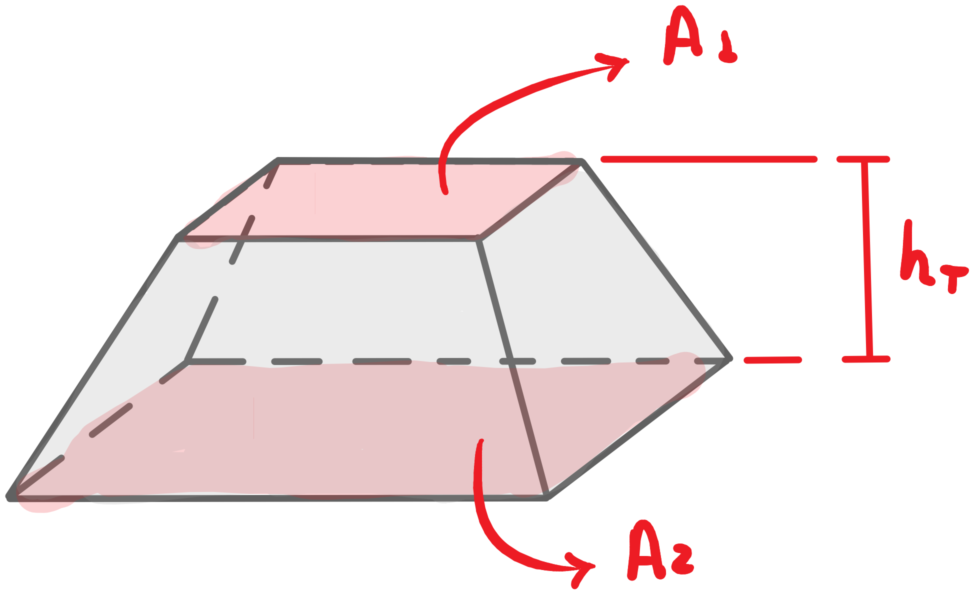
ÁREA :

$$\boxed{\frac{A_b'}{A_b} = \frac{A_L'}{A_L} = \frac{A_T'}{A_T} = K^2}$$

VOLUME :

$$\boxed{\frac{V'}{V} = K^3}$$

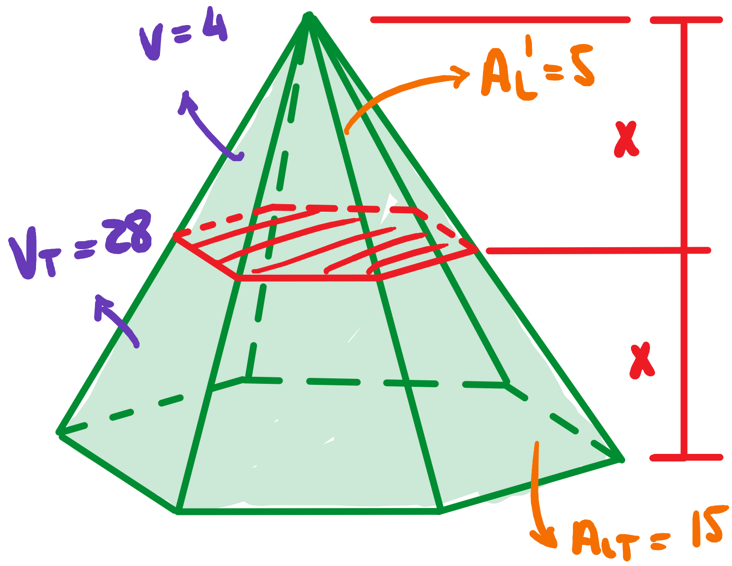
OBS : $\boxed{V_{TRONCO} = V - V'}$




$$\boxed{V_{TRONCO} = \frac{h}{3}\left(A_1 + \sqrt{A_1 \cdot A_2} + A_2\right)}$$

$V_{TOTAL} = 32$  $V_{TRONCO} = ?$

$A_{LAT.} = 20$  $A_{LAT.\ TRONCO} = ?$




$h = 2x$ ; $h' = x$

$$\frac{h'}{h} = \frac{x}{2x} = \frac{1}{2}$$

$$\boxed{K = \frac{1}{2}}$$

$$\frac{V'}{V} = K^3 \rightarrow \frac{V'}{32} = \left(\frac{1}{2}\right)^3 \rightarrow \boxed{V' = 4}$$

$$\boxed{V_{TRONCO} = 32 - 4 = 28}$$

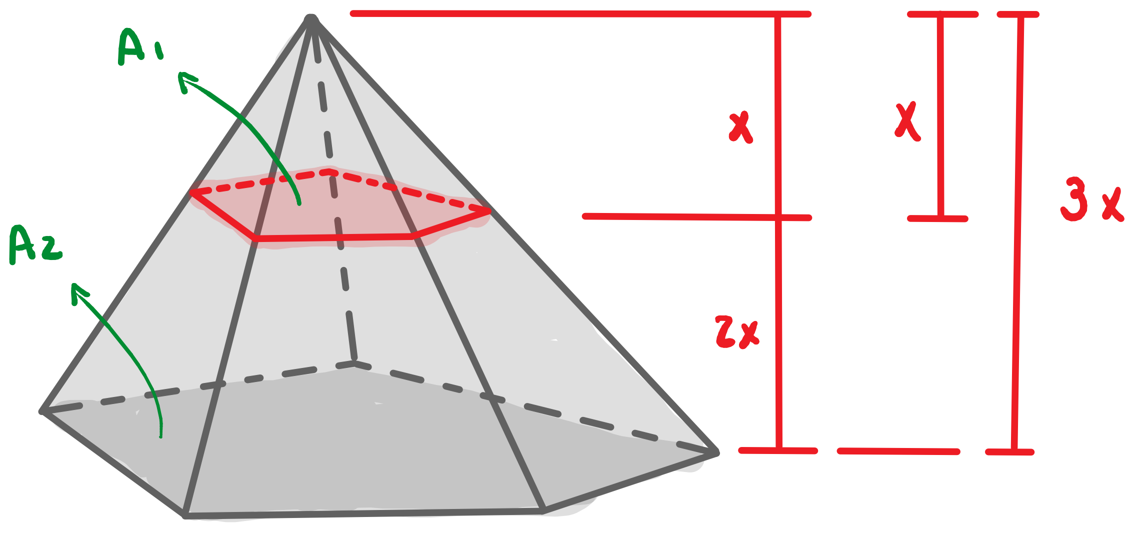
$$\frac{A'_{LAT}}{A_{LAT}} = K^2 \rightarrow \frac{A'_{LAT}}{20} = \left(\frac{1}{2}\right)^2 \rightarrow \underline{A'_{LAT} = 5}$$

$$A_{LAT.\ TRONCO} = A_L - A_L' = 20 - 5 \ ; \ \boxed{A_{LAT\ TRONCO} = 15}$$

## EXEMPLO

UMA PIRÂMIDE É SECCIONADA POR UM PLANO PARALELO A BASE. A DISTÂNCIA DO PLANO À BASE É O DOBRO DA DISTÂNCIA DO PLANO AO VERTICE. CALCULE A RAZÃO ENTRE:

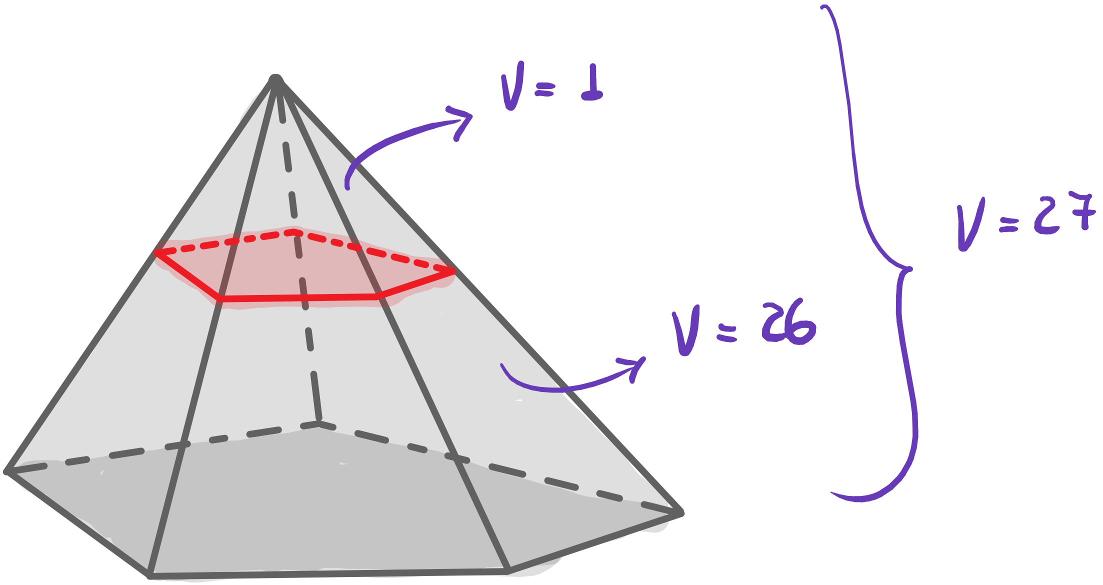
a. ÁREA DAS BASES DO TRONCO
b. VOLUME DO TRONCO E VOLUME TOTAL.




$$K = \frac{x}{3x} \rightarrow \boxed{K = \frac{1}{3}}$$

a) $\dfrac{A_1}{A_2} = K^2 \rightarrow \dfrac{A_1}{A_2} = \left(\dfrac{1}{3}\right)^2 \rightarrow \boxed{\dfrac{A_1}{A_2} = \dfrac{1}{9}}$

b) $\dfrac{V_1}{V_2} = K^3 \rightarrow \dfrac{V_1}{V_2} = \dfrac{1}{27} \rightarrow V_1 = \dfrac{V_2}{27}$




$$\boxed{\dfrac{V_{TRONCO}}{V_{TOTAL}} = \dfrac{26}{27}}$$

$V_1 + V_{TRONCO} = V_2$

$\dfrac{V_2}{27} + V_{TRONCO} = V_2$

$V_{TRONCO} = V_2 - \dfrac{V_2}{27}$

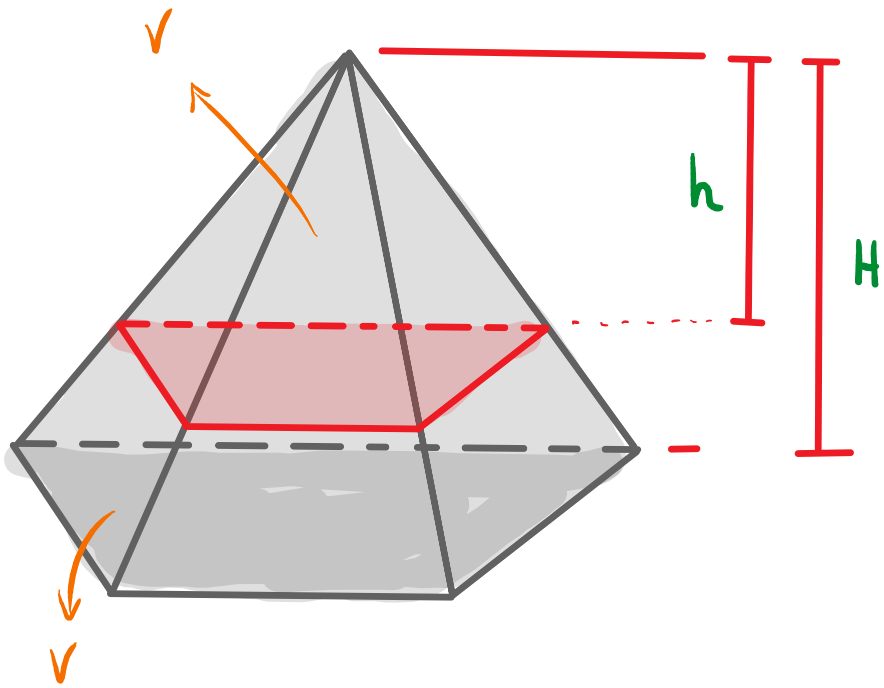
$V_{TRONCO} = \dfrac{26}{27} \cdot V_2$

$\dfrac{V_{TRONCO}}{V_{TOTAL}} = \dfrac{26}{27}$

## EXEMPLO

UM PLANO PARALELO À BASE DE UMA PIRÂMIDE A SECCIONA DIVIDINDO-A EM SÓLIDOS DE MESMO VOLUME.

CALCULE A ALTURA DO TRONCO, SABENDO QUE A ALTURA DA PIRÂMIDE É 12.



$$\left.\begin{array}{l} V_{P.PEQ} = V \\ V_{P.GRANDE} = 2V \end{array}\right\} \quad \frac{V_{PEQ}}{V_{GR}} = K^3$$

$$\frac{\cancel{V}}{2\cancel{V}} = K^3$$

$$K^3 = \frac{1}{2}$$

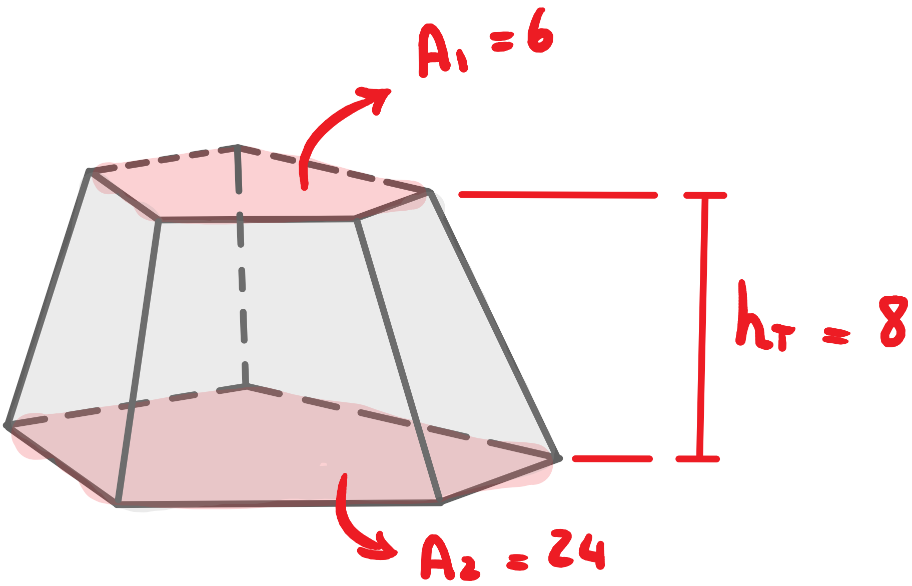
$$K = \sqrt[3]{\frac{1}{2}} = \frac{1}{\sqrt[3]{2}} \cdot \frac{\sqrt[3]{2^2}}{\sqrt[3]{2^2}} = \frac{\sqrt[3]{4}}{2}$$

$$\frac{h}{H} = K \rightarrow h = 12 \cdot \frac{\sqrt[3]{4}}{2}$$

$$\boxed{h = 6\sqrt[3]{4}}$$

## EXEMPLO

CALCULE O VOLUME DO TRONCO DE PIRÂMIDE ABAIXO.




$$V_{TRONCO} = \frac{h}{3}\left(A_1 + \sqrt{A_1 . A_2} + A_2\right)$$

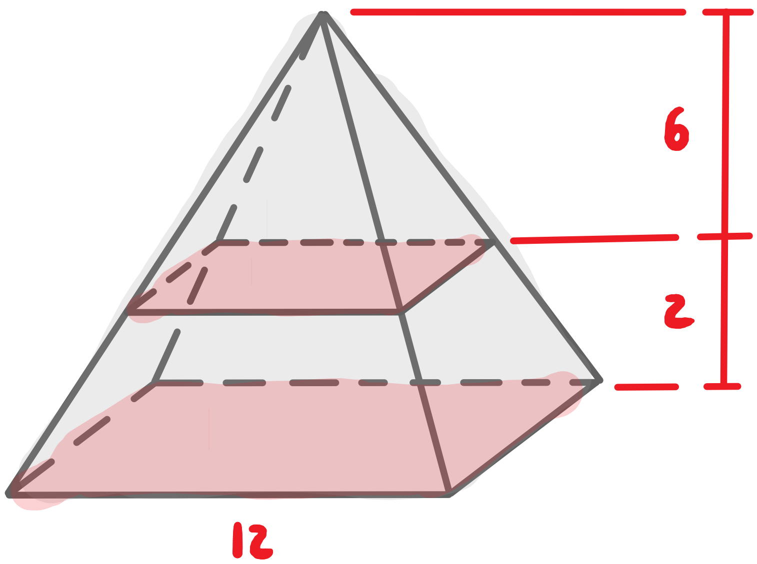
$$V_T = \frac{8}{3}\left(6 + \sqrt{6.24} + 24\right)$$

$$V_T = \frac{8}{3}\left(6 + 12 + 24\right)$$

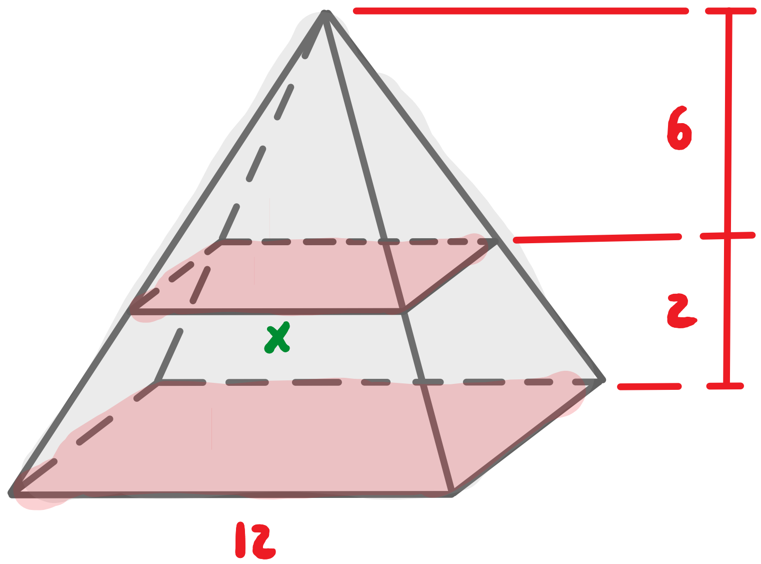
$$V_T = \frac{8}{\cancel{3}} \cdot \overset{14}{\cancel{42}} \rightarrow \boxed{V = 112}$$

# EXEMPLO

CALCULE O VOLUME DO TRONCO GERADO A PARTIR DA PIRÂMIDE REGULAR ABAIXO.



# Solução 1




$$K = \frac{6}{8} = \frac{3}{4}$$

$$\frac{x}{12} = K \rightarrow \frac{x}{12} = \frac{3}{4} \rightarrow \boxed{x = 9}$$

$$A_1 = 9^2 = 81 \quad ; \quad A_2 = 12^2 = 144$$

$$V = \frac{h}{3}\left(A_1 + \sqrt{A_1 . A_2} + A_2\right)$$

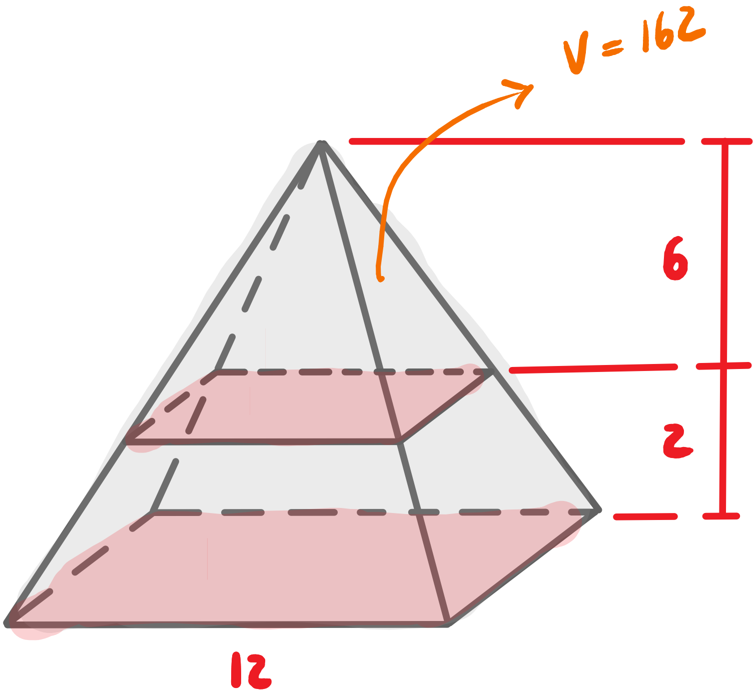
$$V = \frac{2}{3}\left(81 + \sqrt{81 . 144} + 144\right)$$

$$V = \frac{2}{3}\left(81 + 9 . 12 + 144\right)$$

$$V = \frac{2}{3} . 333$$

$$\boxed{V = 222}$$

# Solução 2




$$K = \frac{3}{4}$$

$$V_{TOTAL} = \frac{1}{3} \cdot 12^2 \cdot 8 \rightarrow \underline{V_{TOTAL} = 384}$$

$$\frac{V_P}{V_{TOTAL}} = K^3 \rightarrow V_P = 384 \cdot \left(\frac{3}{4}\right)^3$$

$$V_P = \cancel{384}^{\,6} \cdot \frac{27}{\cancel{64}} \rightarrow \underline{V_P = 162}$$

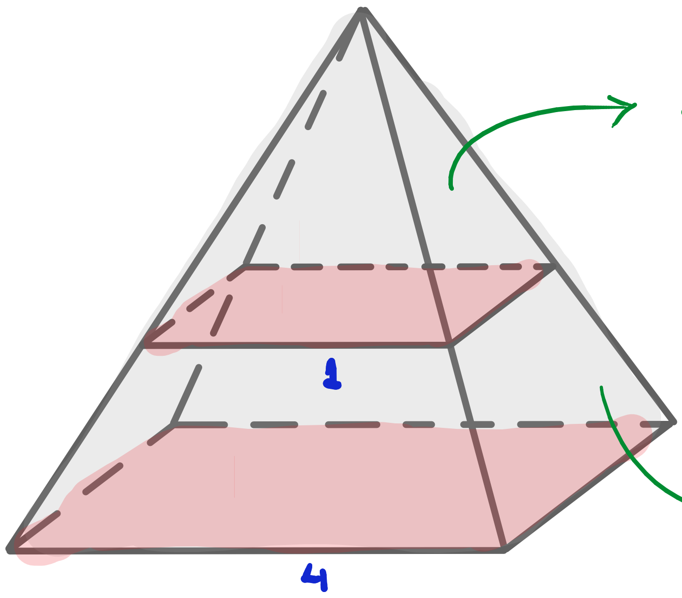
$$V_{TRONCO} = V_{TOTAL} - V_P$$

$$V_{TRONCO} = 384 - 162$$

$$\boxed{V_{TRONCO} = 222}$$

# EXEMPLO

SEJA UM TRONCO DE PIRÂMIDE COM BASES QUADRADAS DE LADO 1 E 4. SE O TRONCO POSSUI VOLUME 126, CALCULE A ALTURA DA PIRÂMIDE QUE DEU ORIGEM AO TRONCO.




1 (2)

$K = \frac{1}{4}$

$V_G = 128$

63 (126)

$$\frac{V_P}{V_G} = K^3 = \frac{1}{64} \longrightarrow V_G = 128$$

$$V_G = \frac{1}{3} \cdot A_b \cdot h \longrightarrow \overset{8}{\cancel{128}} = \frac{1}{3} \cdot \cancel{16} \cdot h$$

$$\boxed{h = 24}$$