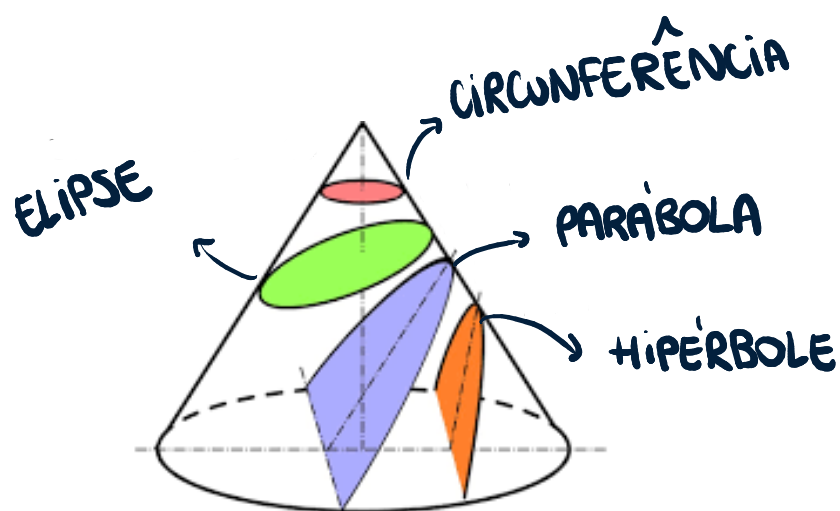
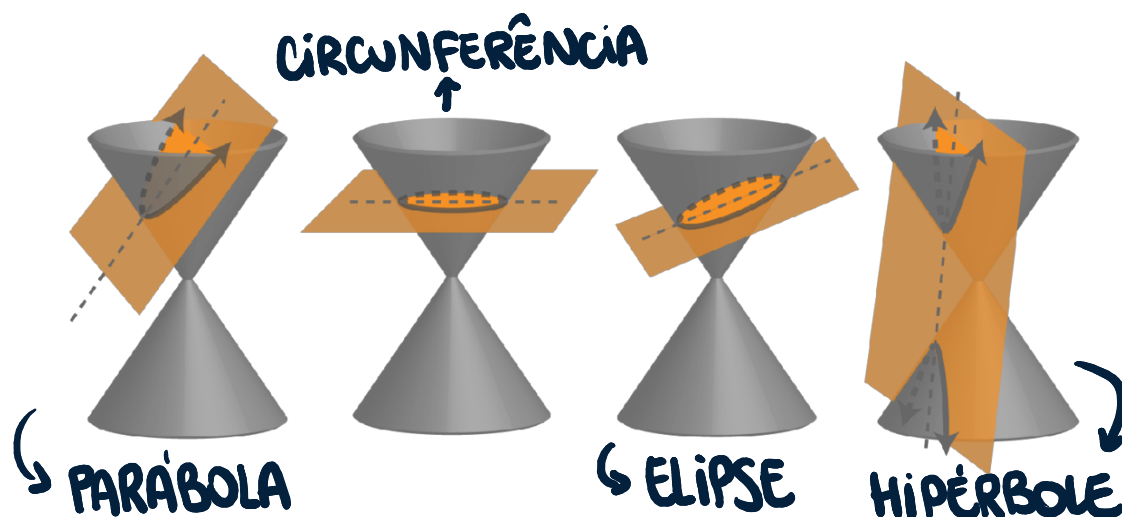


· CÔNICAS ·

SECÇÕES DE UM CONE



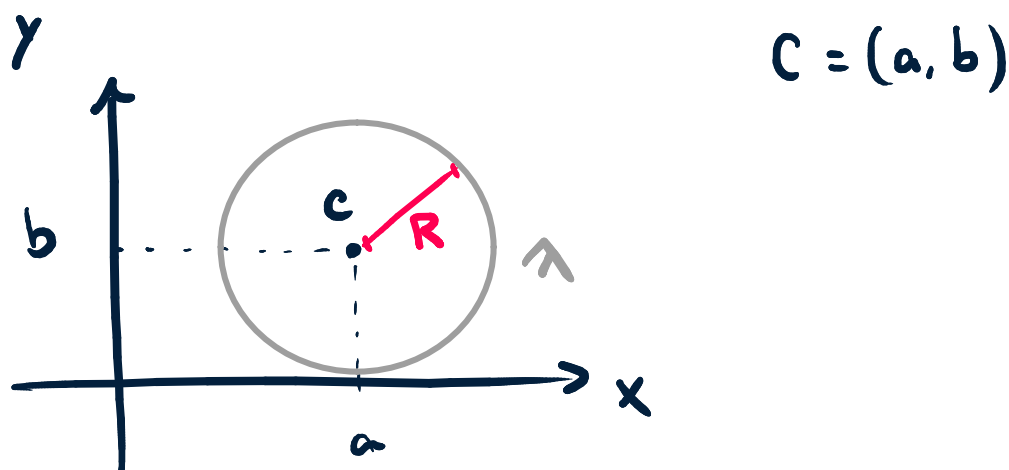
↳ Toda cônica pode ser descrita por uma equação quadrática em x e y :

$$ax^2 + by^2 + cx \cdot y + d \cdot x + e \cdot y + f = 0$$



#01 : CIRCUNFERÊNCIA

EQUAÇÃO REDUZIDA

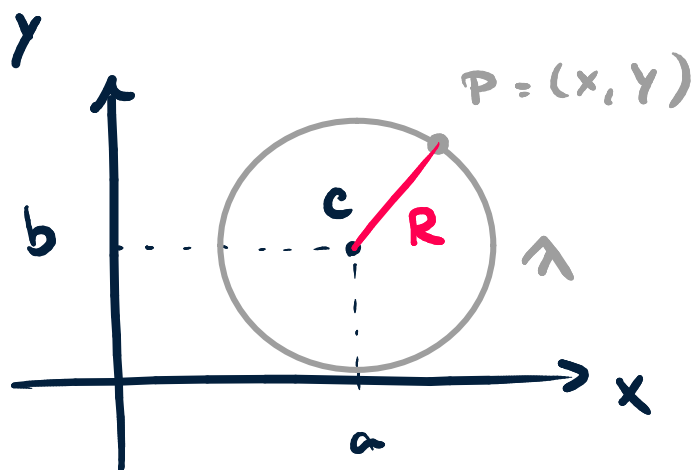


A equação reduzida da circunferência λ é:

$$(x - a)^2 + (y - b)^2 = R^2$$



Prova:



A circunferência de centro $C = (a, b)$ e raio R é o conjunto de todos os pontos $P = (x, y)$ de $\mathbb{R}^2$ que distam R de C :

$$d(P, C) = R$$

$$\sqrt{\Delta x^2 + \Delta y^2} = R$$

$$\sqrt{(x-a)^2 + (y-b)^2} = R$$

$$(x-a)^2 + (y-b)^2 = R^2$$

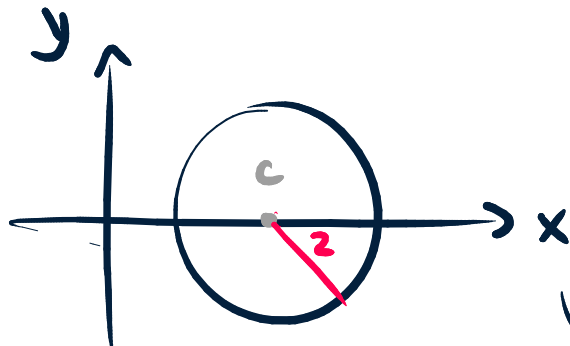
□



EXEMPLO 01

Determine a equação reduzida da circunferência:

a)

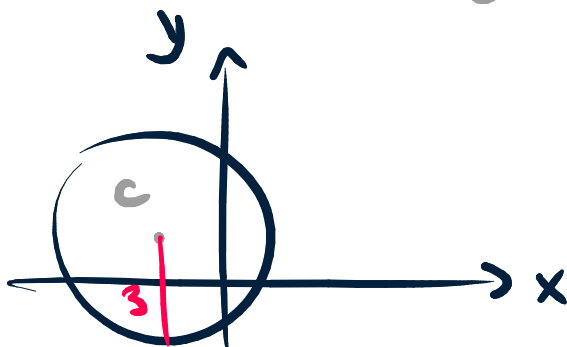


$$C = (3, 0) = (a, b)$$

$$(x - a)^2 + (y - b)^2 = R^2$$

$$(x - 3)^2 + y^2 = 4$$

b)



$$C = (-2, 1) = (a, b)$$

$$(x - a)^2 + (y - b)^2 = R^2$$

$$(x + 2)^2 + (y - 1)^2 = 9$$



EXEMPLO 02

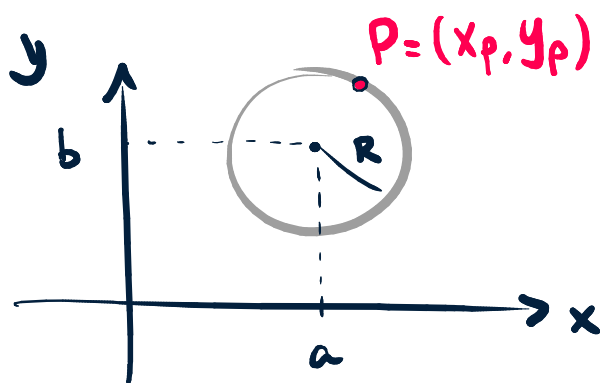
O ponto $P = (1, 4)$ pertence à circunferência?

a) De centro $(1, 1)$ e raio 3?

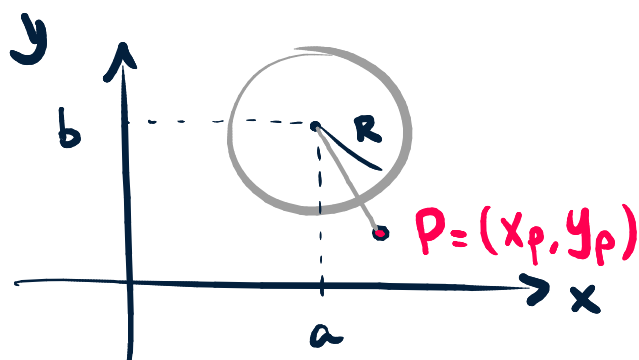
b) De centro $(1, 2)$ e raio 1?

c) De centro $(1, 1)$ e que contém o ponto $(4, 5)$?

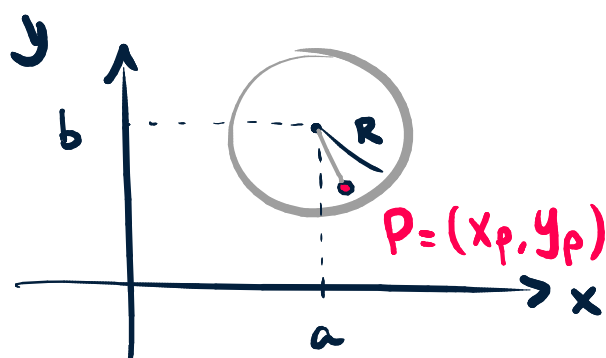
R :



$$(x_p - a)^2 + (y_p - b)^2 = R^2$$



$$(x_p - a)^2 + (y_p - b)^2 > R^2$$



$$(x_p - a)^2 + (y_p - b)^2 < R^2$$



a) De centro $(1,1)$ e raio 3 ? ✓

$$(x-a)^2 + (y-b)^2 = R^2$$

$$(x-1)^2 + (y-1)^2 = 9$$

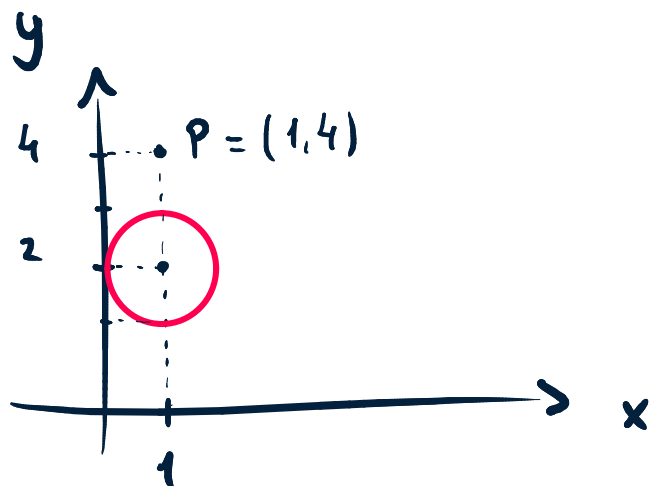
↳ $P = (1,4) : (x-1)^2 + (y-1)^2 = 0^2 + 3^2 = 9 //$

b) De centro $(1,2)$ e raio 1 ? ✗

$$(x-a)^2 + (y-b)^2 = R^2$$

$$(x-1)^2 + (y-2)^2 = 1$$

↳ $P = (1,4) : (x-1)^2 + (y-2)^2 = 0^2 + 2^2 = 4 //$



c) De centro $(1, 1)$ e que contém o ponto $(4, 5)$?



$$(x-1)^2 + (y-1)^2 = R^2$$

→ $P = (4, 5) : (4-1)^2 + (5-1)^2 = R^2$

$$3^2 + 4^2 = R^2$$

$$R = 5$$

$$(x-1)^2 + (y-1)^2 = 5^2$$

→ $P = (1, 4) : (x-1)^2 + (y-1)^2 = 0^2 + 3^2 = 3^2$

interno à circunferência



EQUAÇÃO GERAL

Desenvolvendo a equação reduzida chegamos em:

$$(x - a)^2 + (y - b)^2 = R^2$$

$$x^2 - 2ax + a^2 + y^2 - 2by + b^2 = R^2$$

$$x^2 + y^2 - 2ax - 2by + (a^2 + b^2 - R^2) = 0$$



EXEMPLO

$$2x^2 + 2y^2 - 20x + 8y + 40 = 0$$

$$\hookrightarrow x^2 + y^2 - 10x + 4y + 20 = 0$$

$$x^2 - 10x + \boxed{} + y^2 + 4y + \boxed{} + 20 = 0$$

$$x^2 - 2 \cdot 5 \cdot x + \boxed{5^2} + y^2 + 2 \cdot 2 \cdot y + \boxed{2^2} + 20 = 0 + 5^2 + 2^2$$

$$(x-5)^2 + (y+2)^2 = 3^2$$

$$\hookrightarrow C = (5, -2) \text{ e } R = 3$$



Problema : quando a equação do tipo

$$Ax^2 + By^2 + Cx \cdot y + D \cdot x + E \cdot y + F = 0$$

ira representar uma circunferência ?

↪ Circunferência :

$$Ax^2 + By^2 + Cx \cdot y + D \cdot x + E \cdot y + F = 0 \quad \downarrow \div A$$

$$x^2 + \frac{B}{A}y^2 + \frac{C}{A}x \cdot y + \frac{D}{A}x + \frac{E}{A}y + \frac{F}{A} = 0$$

$$x^2 + y^2 - 2ax - 2by + (a^2 + b^2 - R^2) = 0$$

(i) $A = B$ ✓

$$\hookrightarrow \frac{D}{A} = -2a \therefore a = \frac{-D}{2A} ; \frac{E}{A} = -2b \therefore b = \frac{-E}{2A}$$

(ii) $C = 0$ ✓

$$(iii) \frac{F}{A} = a^2 + b^2 - R^2 = \frac{D^2}{4A^2} + \frac{E^2}{4A^2} - R^2$$

$$\frac{F}{A} = \frac{D^2}{4A^2} + \frac{E^2}{4A^2} - R^2$$

$$\frac{D^2}{4A^2} + \frac{E^2}{4A^2} - \frac{4AF}{4A^2} = R^2$$

$$\frac{D^2 + E^2 - 4AF}{4A^2} = R^2$$

$$D^2 + E^2 - 4AF = R^2 \cdot 4A^2$$

$$D^2 + E^2 - 4AF > 0 \quad \checkmark$$



EXEMPLO

Quais equações representam uma circunferência?

i) $x^2 - y^2 + 4y + 3x + 12 = 0$

ii) $x^2 + y^2 + 4y + 12x - 9 = 0$

iii) $x^2 + y^2 + 3y - 5x + x \cdot y + 10 = 0$

R :

i) $\otimes$ $A = 1$ e $B = -1$: $A \neq B$

ii) $\cdot A = B$ $\checkmark$

$\cdot C = 0$ $\checkmark$

$\cdot D^2 + E^2 - 4AF > 0$

$12^2 + 4^2 - 4 \cdot 1 \cdot (-9) > 0$ $\checkmark$

iii) $\otimes$ $C \neq 0$



EXEMPLO

Encontre o centro e o raio da circunferência :

a) $x^2 + y^2 - 2x - 6y - 6 = 0$

b) $x^2 + y^2 + 10y + 24 = 0$

R :

a) $x^2 - 2x + \boxed{} + y^2 - 6y + \boxed{} - 6 = 0$

$$x^2 - 2x + \boxed{1^2} + y^2 - 6y + \boxed{3^2} - 6 = +1 + 9$$

$$(x-1)^2 + (y-3)^2 = 4^2$$

↪ $C = (1, 3)$ e $R = 4$

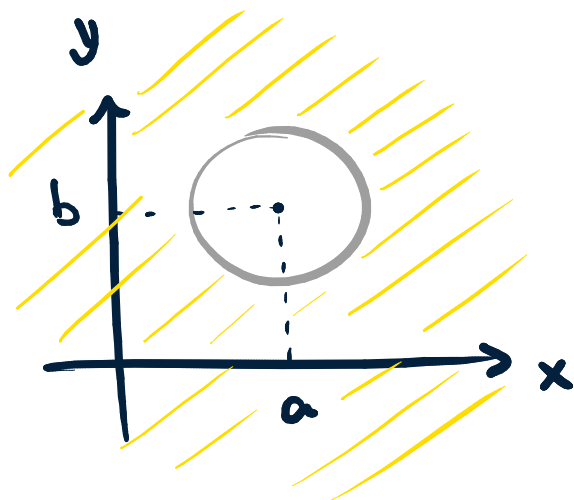
b) $x^2 + y^2 + 10y + \boxed{} + 24 = 0$

$$x^2 + y^2 + 10y + \boxed{} + 24 + 1 = 0 + 1$$

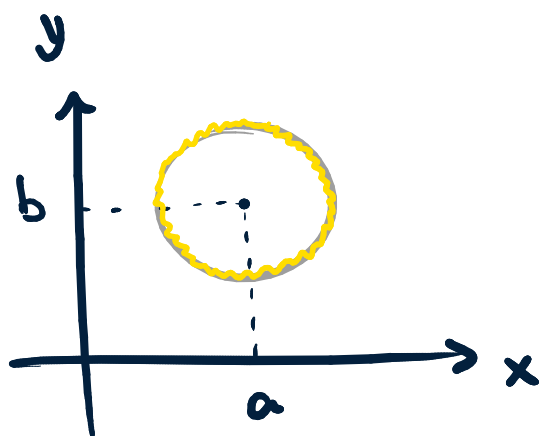
$$(x-0)^2 + (y+5)^2 = 1^2$$

↪ $C = (0, -5)$ e $R = 1$

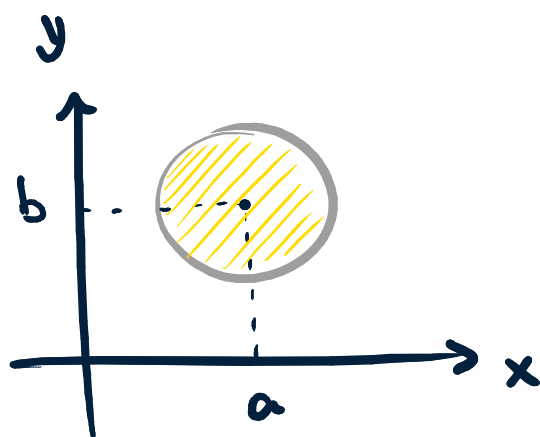
INEQUAÇÕES



$$(x-a)^2 + (y-b)^2 > r^2$$



$$(x-a)^2 + (y-b)^2 = r^2$$



$$(x-a)^2 + (y-b)^2 < r^2$$



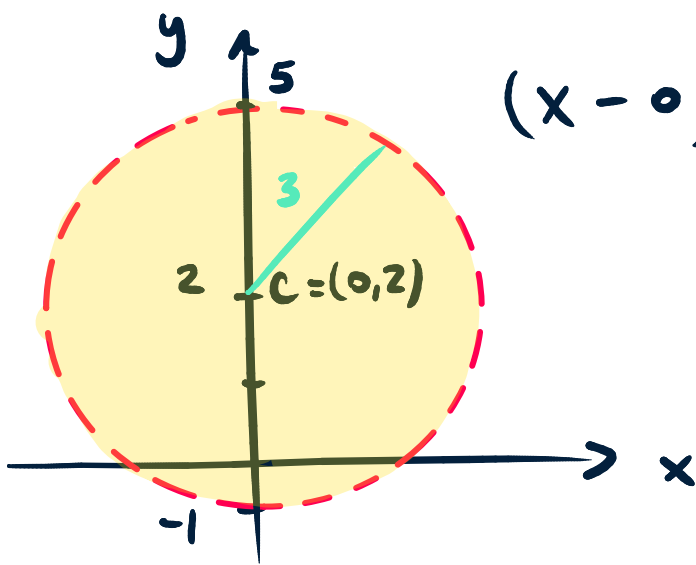
EXEMPLO

Resolva a inequação: $x^2 + y^2 - 4y - 5 < 0$

R: $x^2 + y^2 - 4y + 2^2 - 5 < 0 + 2^2$

$$(x - 0)^2 + (y - 2)^2 < 3^2$$

$\hookrightarrow C = (0, 2) \quad e \quad R = 3$

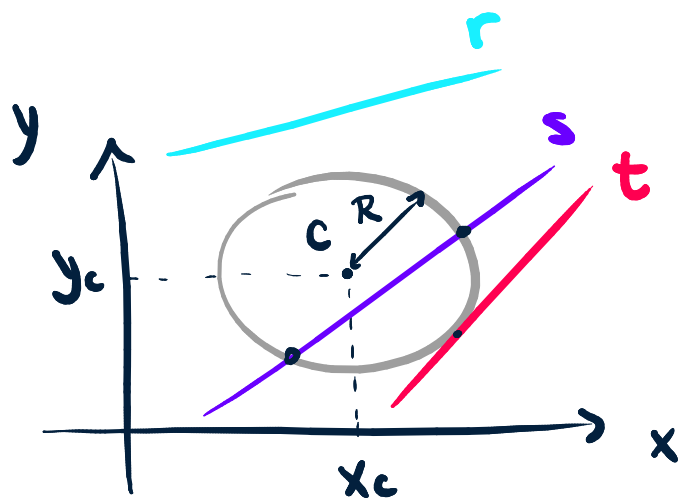


$$(x - 0)^2 + (y - 2)^2 < 3^2$$



RETA e CIRCUNFERÊNCIA

$$C = (x_c, y_c)$$



t : tangente

s : secante

r : exterior



EXEMPLO

Determine a posição da reta $r: x - 2y - 1 = 0$ em relação à circunferência $x^2 + y^2 - 16 = 0$.

R :

SOLUÇÃO 01

$$\begin{cases} x^2 + y^2 - 16 = 0 \\ x - 2y - 1 = 0 \end{cases} \rightarrow \begin{cases} x^2 + y^2 = 16 \\ x = 1 + 2y \end{cases}$$

$$(1 + 2y)^2 + y^2 = 16 \quad \therefore 4y^2 + 8y + 1 + y^2 - 16 = 0$$

$$5y^2 + 8y - 15 = 0$$

$$\Delta = 8^2 - 4 \cdot 5 \cdot (-15) > 0$$

↗ 2 raízes distintas



reta secante!



SOLUÇÃO 02

$$C = (x_c, y_c) = (0, 0) \text{ e } R = 4$$

$$r: ax + by + c = 0$$

$$r: x - 2y - 1 = 0$$

$$d(C, r) = \frac{|ax_c + by_c + c|}{\sqrt{a^2 + b^2}} = \frac{|1 \cdot 0 - 2 \cdot 0 - 1|}{\sqrt{1^2 + 4}}$$

$$d(C, r) = \frac{1}{\sqrt{5}} \approx 0,5 < R \rightarrow \underline{\text{secante}}$$



Dada a reta $s: ax + by + c = 0$ e a circunferência $(x - x_c)^2 + (y - y_c)^2 = R^2$, temos:

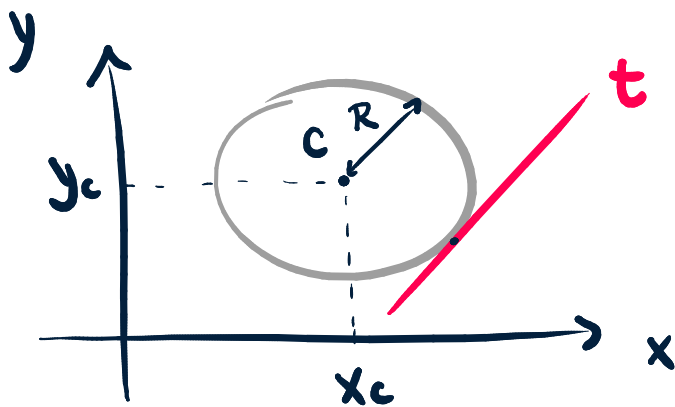
tangente:

• se a eq. quadrática resultante do sistema

$$\begin{cases} (x - x_c)^2 + (y - y_c)^2 = R^2 \\ ax + by + c = 0 \end{cases}$$

tiver $\Delta = 0$

$$\text{ou se } d(C, s) = R = \frac{|ax_c + by_c + c|}{\sqrt{a^2 + b^2}}$$



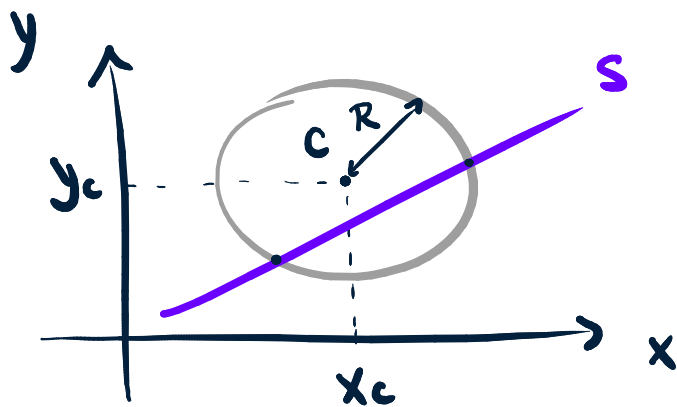
secante:

• se a eq. quadrática resultante do sistema

$$\begin{cases} (x-x_c)^2 + (y-y_c)^2 = R^2 \\ ax + by + c = 0 \end{cases}$$

tiver $\Delta > 0$

$$\text{ou se } d(C, s) < R = \frac{|ax_c + by_c + c|}{\sqrt{a^2 + b^2}}$$



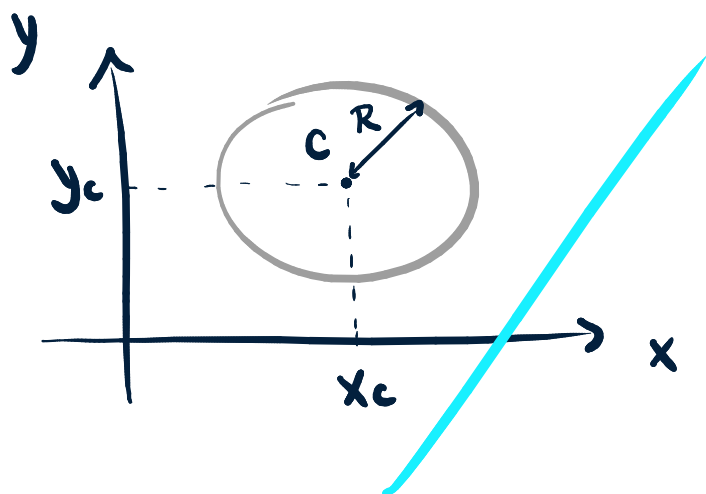
exterior:

• se a eq. quadrática resultante do sistema

$$\begin{cases} (x-x_c)^2 + (y-y_c)^2 = R^2 \\ ax + by + c = 0 \end{cases}$$

tiver $\Delta < 0$

ou se $d(C,s) > R = \frac{|ax_c + by_c + c|}{\sqrt{a^2 + b^2}}$

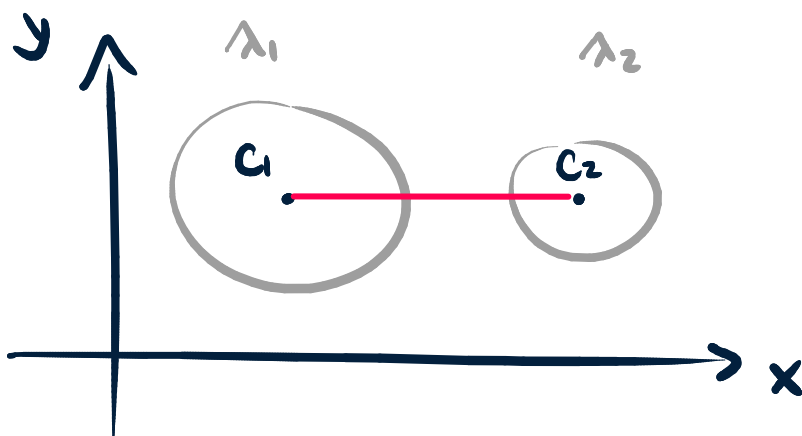


CIRCUNFERÊNCIA: POSIÇÃO RELATIVA

CASO 01 :

CIRCUNFERÊNCIAS EXTERIORES

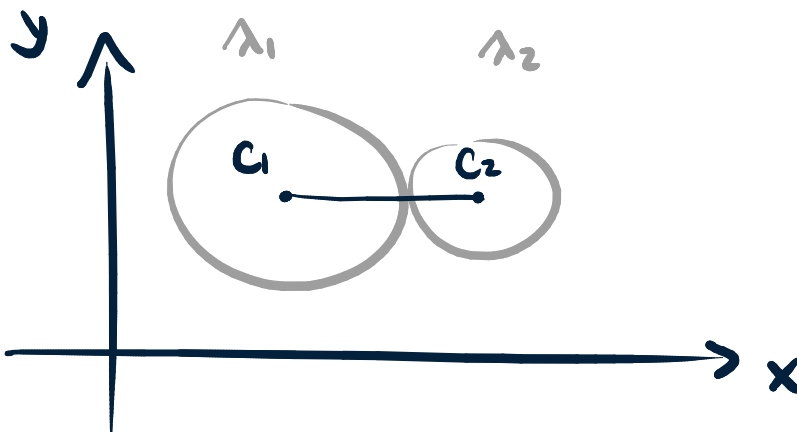
$$d(C_1, C_2) > R_1 + R_2$$



CASO 02 :

CIRCUNFERÊNCIAS TANGENTES EXTERIORMENTE

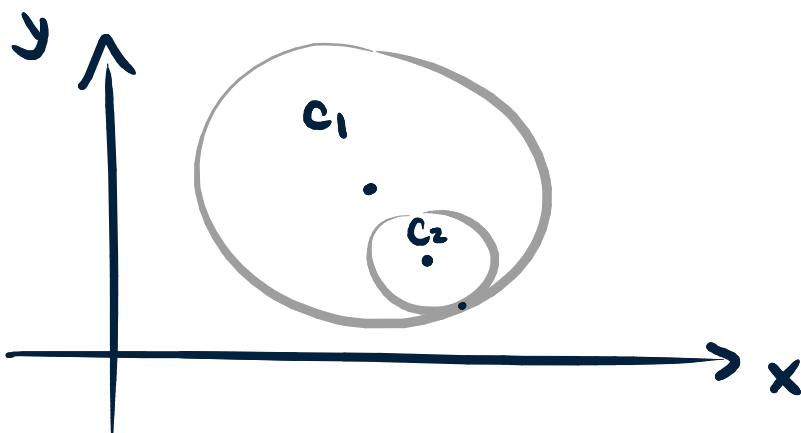
$$d(C_1, C_2) = R_1 + R_2$$



CASO 03 :

CIRCUNFERÊNCIAS TANGENTES INTERIORMENTE

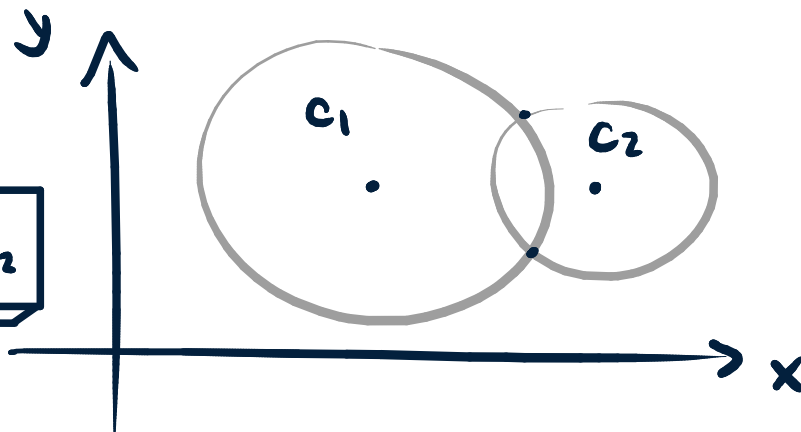
$$d(C_1, C_2) = |R_1 - R_2|$$



CASO 04 :

CIRCUNFERÊNCIAS SECANTES

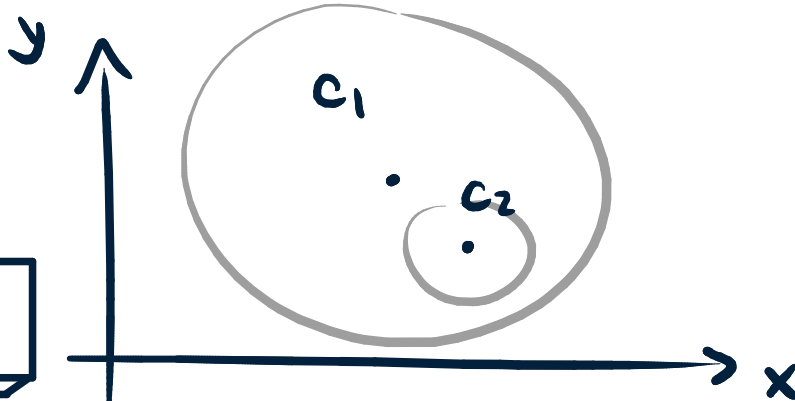
$$|R_1 - R_2| < d(C_1, C_2) < R_1 + R_2$$



CASO 05 :

CIRCUNFERÊNCIA MENOR É INTERIOR À MAIOR

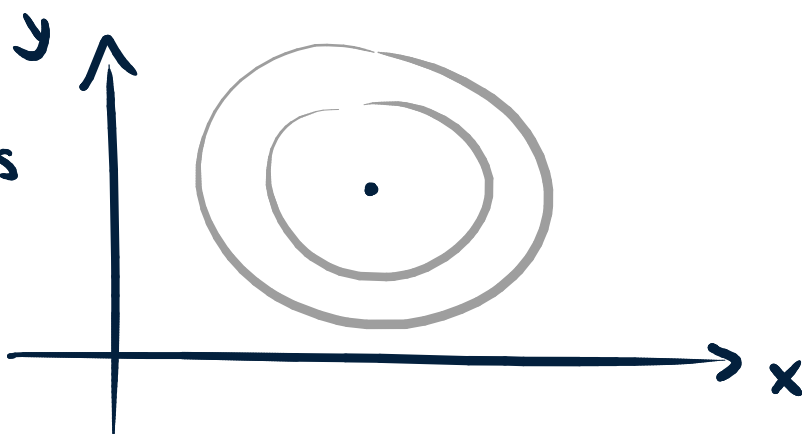
$$0 \leq d(C_1, C_2) < |R_1 - R_2|$$



CASO 06 :

CIRCUNFERÊNCIAS CONCÊNTRICAS

$$d(C_1, C_2) = 0$$



EXEMPLO

Determine a posição relativa das circunferências:

$$\lambda_1: x^2 + y^2 - 2x - 6y - 6 = 0$$

$$\lambda_2: x^2 + y^2 + 10y + 24 = 0$$

R:

$$\lambda_1: \underbrace{x^2 - 2x + 1^2}_{(x-1)^2} + \underbrace{y^2 - 6y + 3^2}_{(y-3)^2} - 6 = 0 + 1 + 9$$
$$(x-1)^2 + (y-3)^2 = 4^2$$

→ $C_1 = (1, 3)$ e $R_1 = 4$

$$\lambda_2: \underbrace{x^2 + y^2 + 10y + 5^2}_{(x-0)^2 + (y+5)^2} + 24 = 0 + 25$$
$$(x-0)^2 + (y+5)^2 = 1^2$$

→ $C_2 = (0, -5)$ e $R_2 = 1$

$$\rightarrow d(C_1, C_2) = \sqrt{\Delta x^2 + \Delta y^2} = \sqrt{1^2 + 64} = \sqrt{65} \approx 8,1$$

$$R_1 + R_2 = 4 + 1 = 5$$



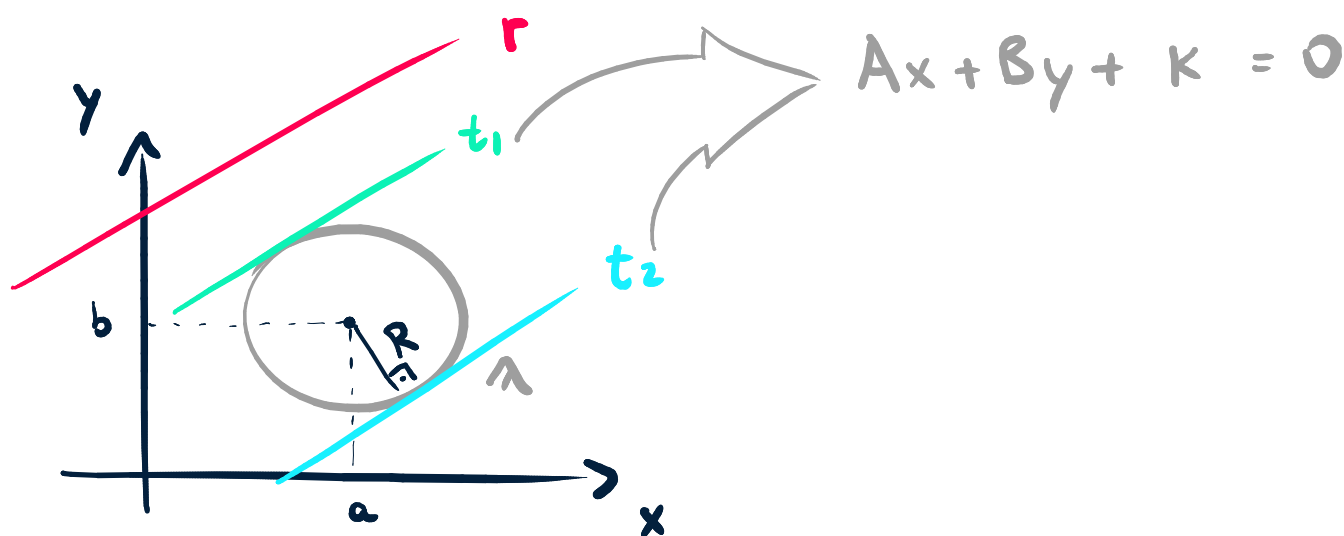
UNIVERSO NARRADO

↓
CIRCUNFERÊNCIAS EXTERIORES

CIRCUNFERÊNCIA: PROBLEMA DE APROFUNDAMENTO #01

Dada a reta $r: Ax + By + C = 0$ e a circunferência $\lambda: (x-a)^2 + (y-b)^2 = R^2$, determine uma reta tangente a λ que seja paralela à reta r .

R:



$$d(t, \text{Centro}) = R \quad \therefore \quad \frac{|Ax_c + By_c + K|}{\sqrt{A^2 + B^2}} = R$$

$$|A \cdot a + B \cdot b + K| = R \cdot \sqrt{A^2 + B^2}$$

$$\text{i) } A \cdot a + B \cdot b + K_1 = R \cdot \sqrt{A^2 + B^2}$$

$$\text{ii) } A \cdot a + B \cdot b + K_2 = -R \cdot \sqrt{A^2 + B^2}$$



CIRCUNFERÊNCIA: PROBLEMA DE APROFUNDAMENTO #02

Dado o ponto $P = (3, 9)$, que pertence à circunferência $\Lambda: (x-2)^2 + (y-5)^2 = 17$, determine uma reta tangente a Λ que passa por P .

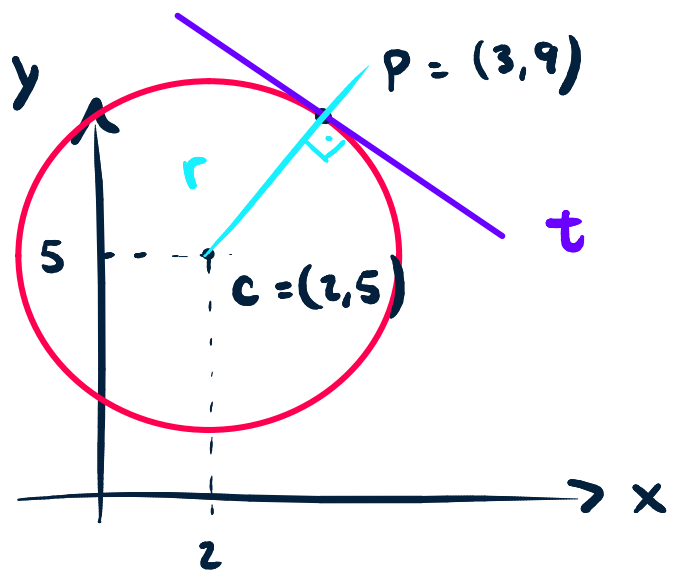
R: $\Lambda: (x-2)^2 + (y-5)^2 = 17$

$C = (2, 5)$ e $R = \sqrt{17}$

i) $t: y = mx + n$
 $9 = m \cdot 3 + n$

$n = 9 - 3m$

$y = mx + (9 - 3m)$



ii) $r: y = ax + b$

$a = \frac{\Delta y}{\Delta x} = \frac{9-5}{3-2} = 4$

$y = 4x + b$
 $5 = 4 \cdot 2 + b \therefore b = -3$

$r: y = 4x - 3$

iii) $m_r \cdot m_s = -1 \therefore$

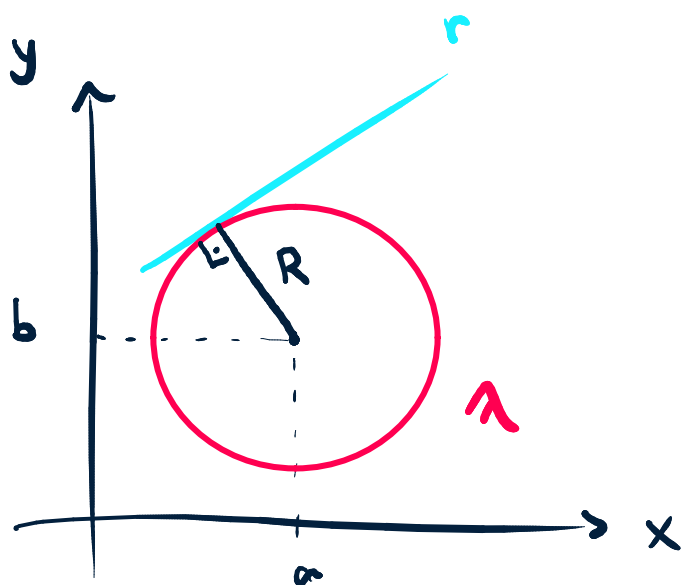
$m_s = -1/4 \therefore$

$y = -\frac{x}{4} + \frac{39}{4}$

CIRCUNFERÊNCIA: PROBLEMA DE APROFUNDAMENTO #03

① Determine a circunferência de centro (a, b) que tangencia a reta $r: Ax + By + C = 0$.

R:



$$d(\text{Centro}, r) = R = \frac{|Ax_c + By_c + C|}{\sqrt{A^2 + B^2}}$$

$$R = \frac{|A \cdot a + B \cdot b + C|}{\sqrt{A^2 + B^2}}$$

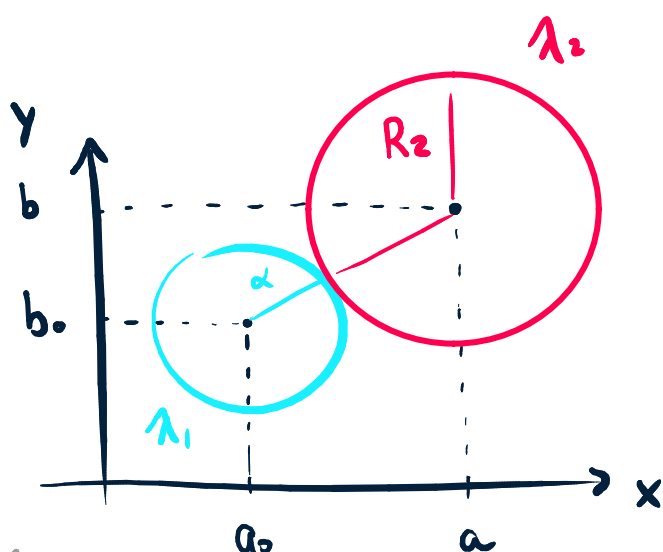
→ $\lambda: (x - a)^2 + (y - b)^2 = R^2$



CIRCUNFERÊNCIA: PROBLEMA DE APROFUNDAMENTO #04

Determine a circunferência de centro (a, b) que tangencia a circunferência $(x - a_0)^2 + (y - b_0)^2 = \alpha^2$

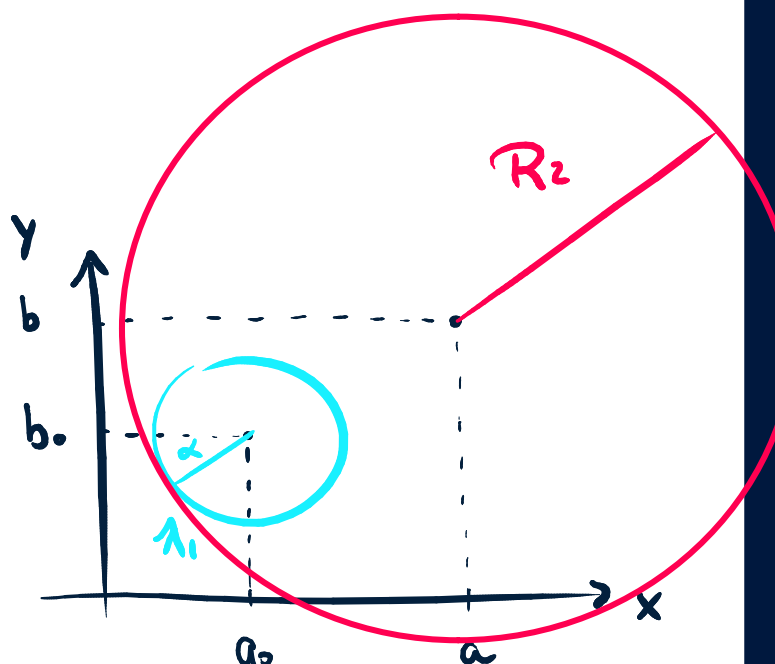
R :



$$\hookrightarrow d(C_1, C_2) = \alpha + R_2$$

$$\sqrt{(a - a_0)^2 + (b - b_0)^2} = \alpha + R_2$$

$$R_2 = \sqrt{(a - a_0)^2 + (b - b_0)^2} - \alpha$$

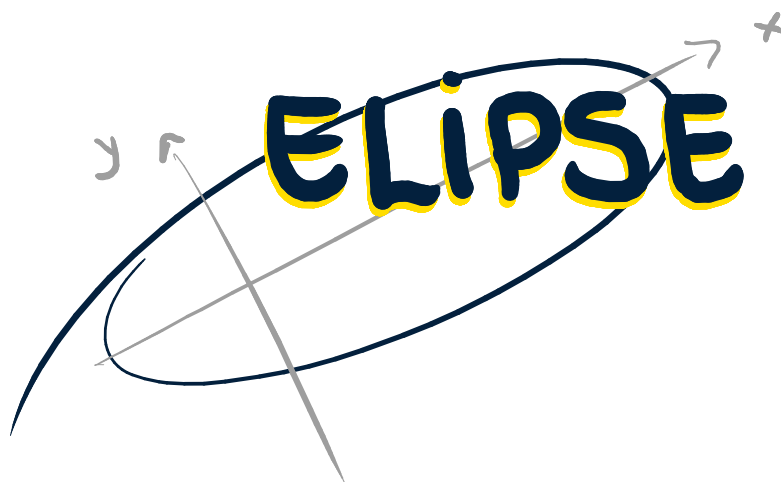


$$\hookrightarrow d(C_1, C_2) = R_2 - \alpha$$

$$\sqrt{(a - a_0)^2 + (b - b_0)^2} = R_2 - \alpha$$

$$R_2 = \sqrt{(a - a_0)^2 + (b - b_0)^2} + \alpha$$

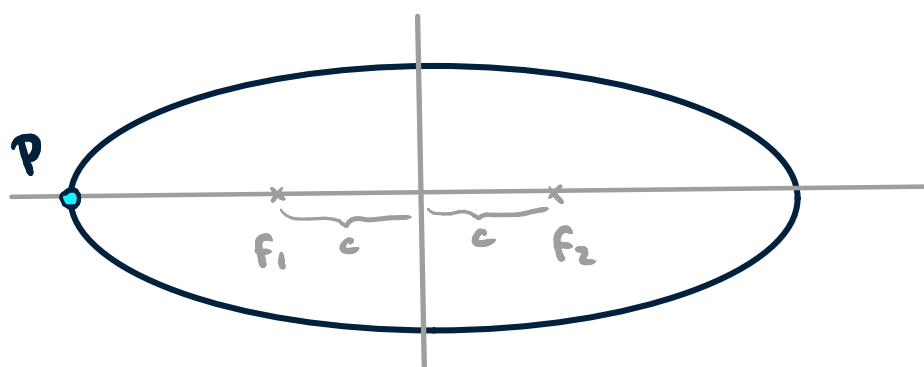
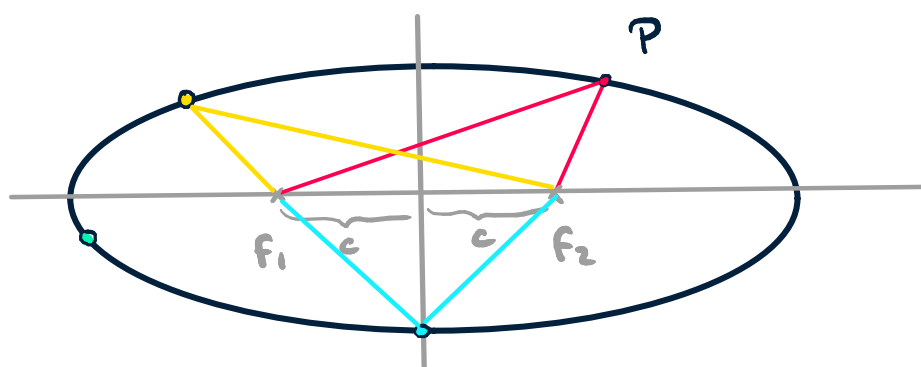




DEFINIÇÃO

Uma elipse é o conjunto dos pontos P tal que

$$\overline{PF_1} + \overline{PF_2} = 2a = \text{constante}.$$

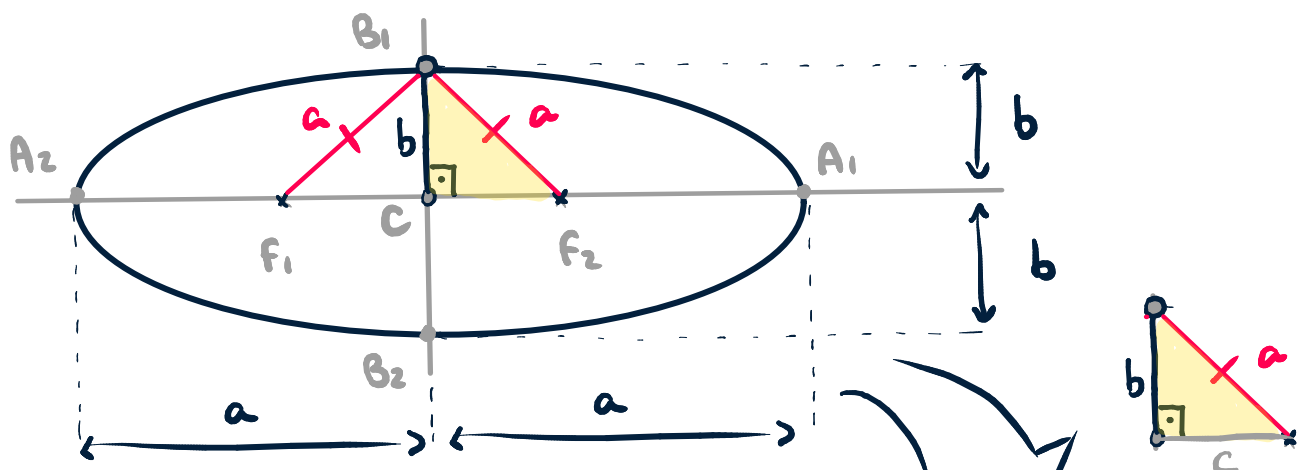


$$\overline{PF_1} + \overline{PF_2} = \text{cte}$$

$$(\cancel{a-c}) + (\cancel{a+c}) = \text{cte}$$

$$\underline{\underline{\text{cte} = 2a}}$$





RELAÇÃO NOTÁVEL

$$a^2 = b^2 + c^2$$

- F_1 e F_2 : focos da elipse
- O : centro
- $\overline{A_1 A_2} = 2a$: eixo maior
- $\overline{B_1 B_2} = 2b$: eixo menor
- $\overline{F_1 F_2} = 2c$: distância focal
- $\frac{c}{a} = e$: excentricidade

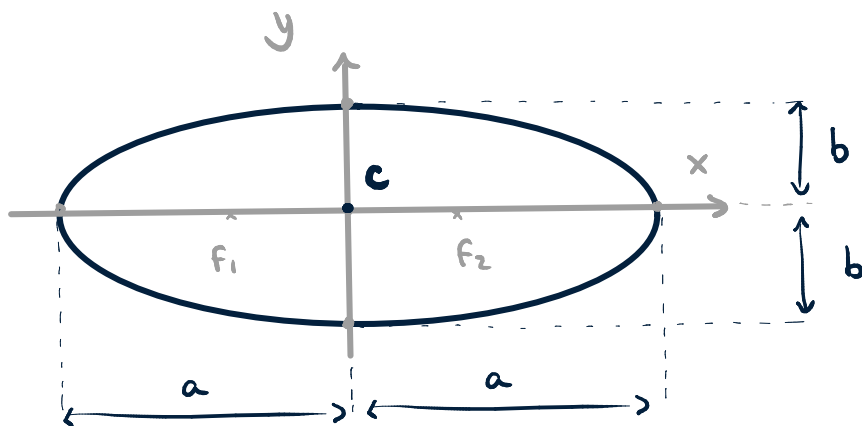


ELIPSE : EQUAÇÃO REDUZIDA

A equação reduzida da elipse de eixo maior $2a$ e eixo menor $2b$, paralelos aos eixos x e y , respectivamente, é:

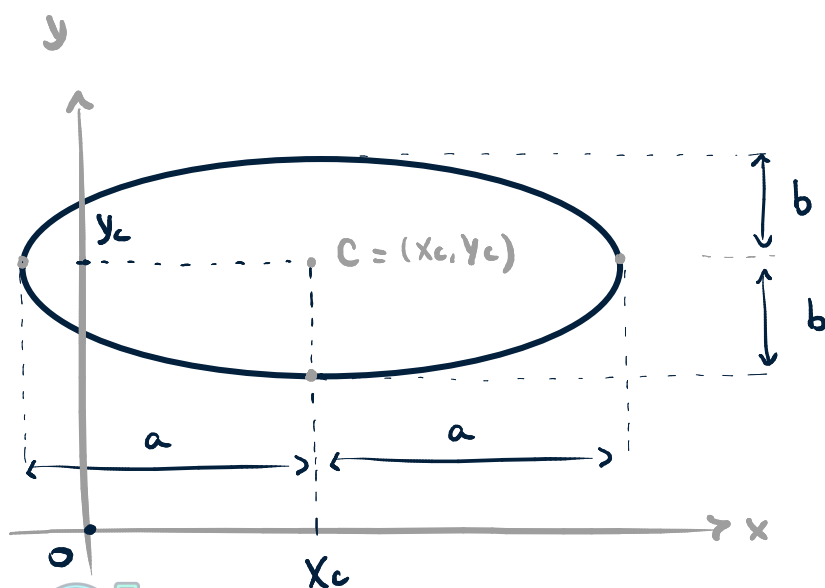
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

centro na origem



$$\frac{(x-x_c)^2}{a^2} + \frac{(y-y_c)^2}{b^2} = 1$$

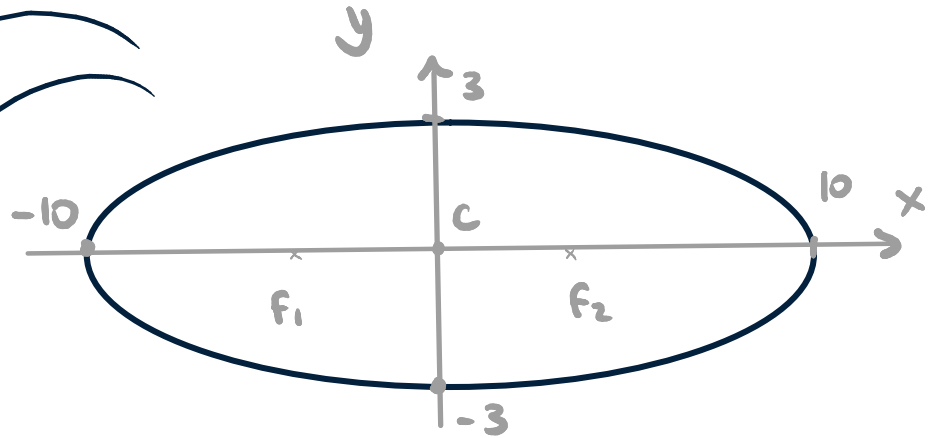
centro em (x_c, y_c)



EXEMPLO

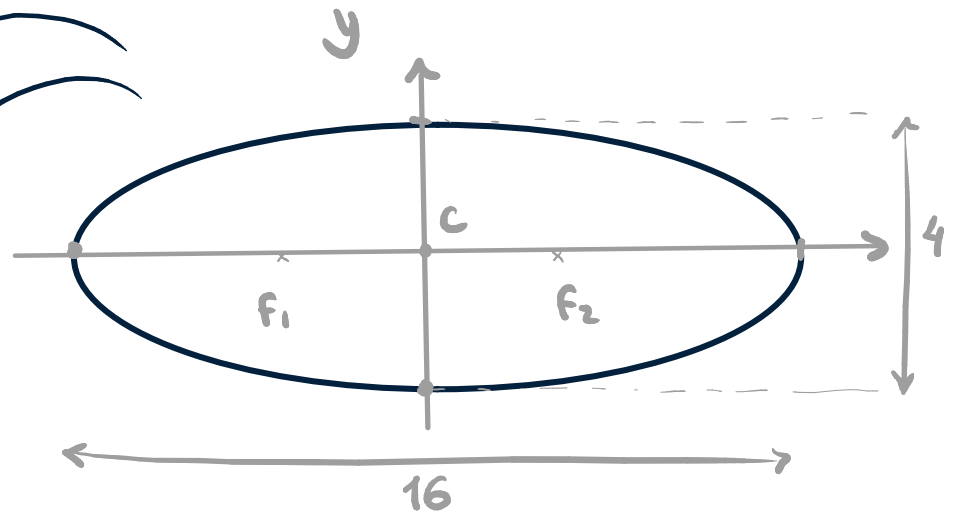
(i)

$$\frac{x^2}{10^2} + \frac{y^2}{3^2} = 1$$



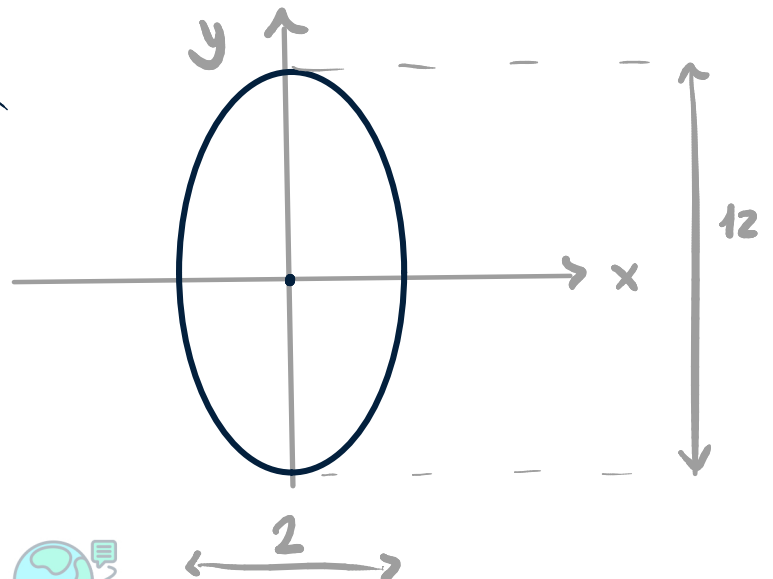
(ii)

$$\frac{x^2}{8^2} + \frac{y^2}{2^2} = 1$$



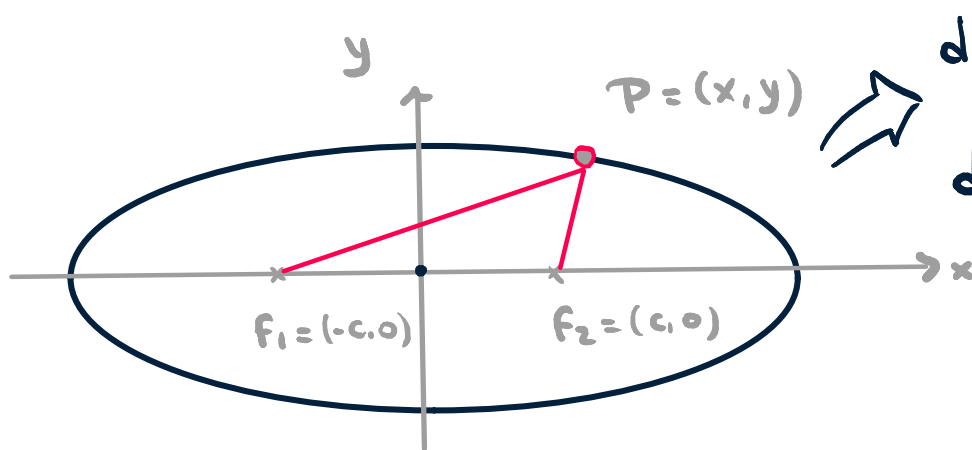
(iii)

$$\frac{x^2}{1^2} + \frac{y^2}{6^2} = 1$$



ELIPSE : EQUAÇÃO REDUZIDA

(DEMONSTRAÇÃO)



$$d(P; F_1) + d(P; F_2) = 2a$$

$$d(A, B) = \sqrt{\Delta x^2 + \Delta y^2}$$

$$d(P; F_1) + d(P; F_2) = 2a$$

$$\sqrt{(x+c)^2 + (y-0)^2} + \sqrt{(x-c)^2 + (y-0)^2} = 2a$$

$$\sqrt{(x+c)^2 + y^2} + \sqrt{(x-c)^2 + y^2} = 2a$$

$$\left(\sqrt{(x+c)^2 + y^2} \right)^2 = \left(2a - \sqrt{(x-c)^2 + y^2} \right)^2$$

$$(x+c)^2 + y^2 = 4a^2 + (x-c)^2 + y^2 - 4a\sqrt{(x-c)^2 + y^2}$$

$$\cancel{x^2} + \cancel{c^2} + 2xc + \cancel{y^2} = 4a^2 + \cancel{x^2} + \cancel{c^2} - 2xc + \cancel{y^2} - 4a\sqrt{(x-c)^2 + y^2}$$

$$4xc - 4a^2 = -4a\sqrt{(x-c)^2 + y^2}$$

$$xc - a^2 = -a\sqrt{(x-c)^2 + y^2}$$



$$(xc - a^2)^2 = \left(-a\sqrt{(x-c)^2 + y^2} \right)^2$$

$$x^2c^2 + a^4 - 2xca^2 = a^2[(x-c)^2 + y^2]$$

$$x^2c^2 + a^4 - 2xca^2 = a^2[x^2 - 2xc + c^2 + y^2]$$

$$x^2c^2 + a^4 - 2xca^2 = a^2x^2 - 2xca^2 + a^2c^2 + a^2y^2$$

$$a^2x^2 - x^2c^2 + a^2c^2 + a^2y^2 = a^4$$

$$x^2(\underbrace{a^2 - c^2}_{b^2}) + y^2a^2 = a^2(\underbrace{a^2 - c^2}_{b^2})$$

$$x^2 \cdot b^2 + y^2 a^2 = a^2 \cdot b^2$$

↘ $\div a^2 b^2$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad \square$$



EXEMPLO 01

Esboce a elipse

$$i) \frac{x^2}{25} + \frac{y^2}{16} = 1$$

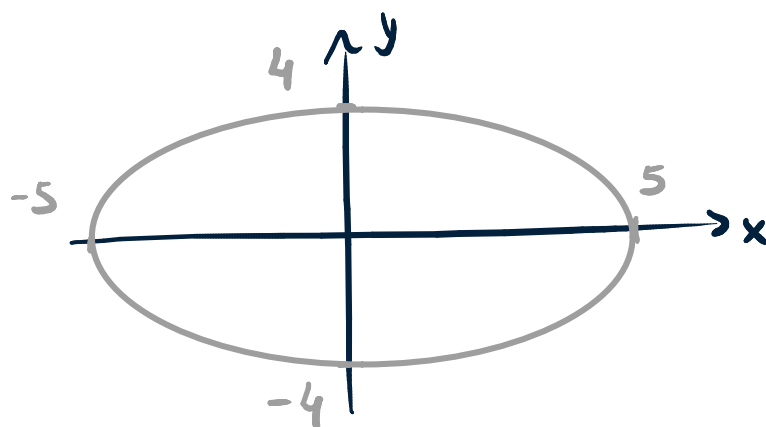
$$ii) \frac{(x-1)^2}{100} + \frac{(y+3)^2}{400} = 1$$

R :

$$i) \frac{x^2}{5^2} + \frac{y^2}{4^2} = 1$$

$$a = 5$$
$$2a = 10$$

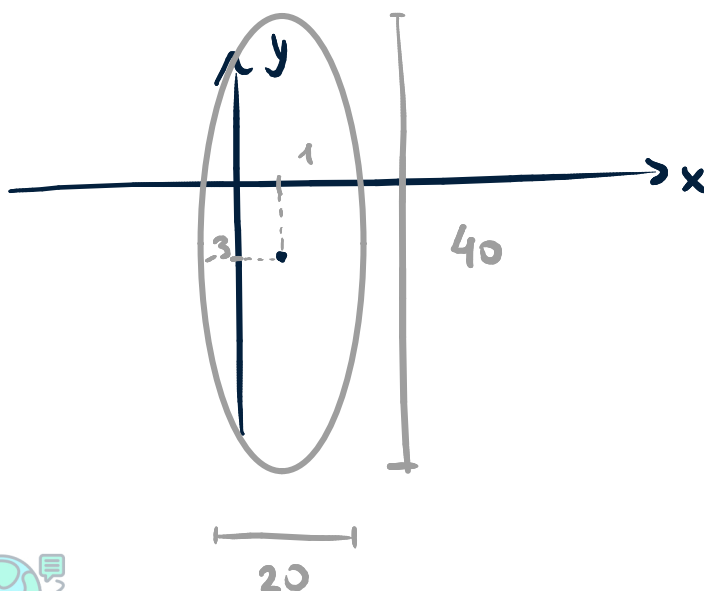
$$b = 4$$
$$2b = 8$$



$$ii) \frac{(x-1)^2}{10^2} + \frac{(y+3)^2}{20^2} = 1$$

$$b = 10$$
$$2b = 20$$

$$a = 20$$
$$2a = 40$$



$$C = (1, -3)$$

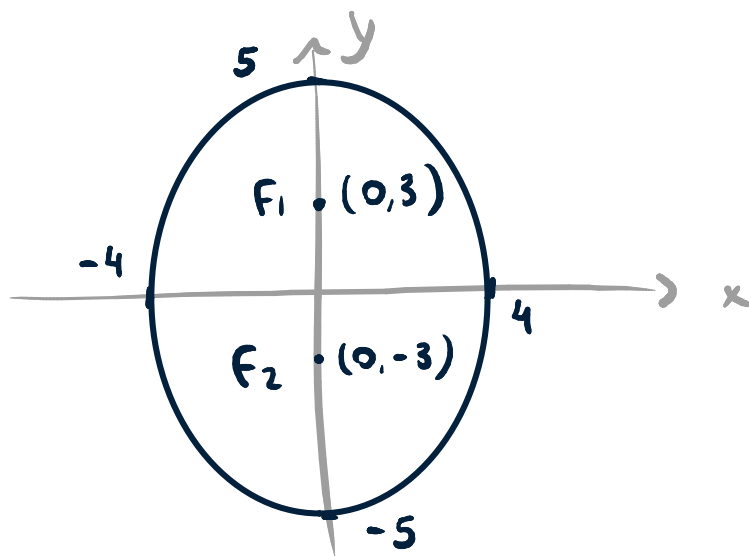


EXEMPLO 02

Uma elipse têm os focos em $(0, 3)$ e $(0, -3)$.

Se o eixo menor vale 8, determine sua equação.

R :



$$\begin{aligned} & \cdot 2b = 8 \therefore b = 4 \\ & \cdot 2c = 6 \therefore c = 3 \end{aligned} \quad \left. \vphantom{\begin{aligned} & \cdot 2b = 8 \therefore b = 4 \\ & \cdot 2c = 6 \therefore c = 3 \end{aligned}} \right\} \begin{aligned} & a^2 = b^2 + c^2 = 4^2 + 3^2 \\ & a = 5 \end{aligned}$$
$$\cdot 2a = 10 \therefore a = 5$$

↪

$$\boxed{\frac{x^2}{4^2} + \frac{y^2}{5^2} = 1}$$



EXEMPLO 03

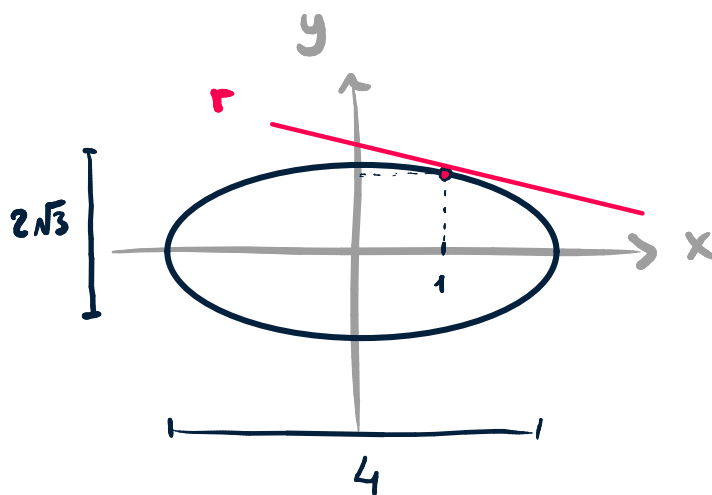
Uma elipse tem equação dada por $\frac{x^2}{4} + \frac{y^2}{3} = 1$.

Determine a reta que tangencia a elipse no ponto $(1, \frac{3}{2})$.

R : $\frac{x^2}{2^2} + \frac{y^2}{\sqrt{3}^2} = 1$.

$a = 2$
 $2a = 4$

$b = \sqrt{3}$
 $2b = 2\sqrt{3}$



$$\text{i) } \left. \begin{array}{l} r : y = mx + n \\ \frac{3}{2} = m \cdot 1 + n \end{array} \right\} n = \frac{3}{2} - m$$

$$r : y = mx + \left(\frac{3}{2} - m \right)$$

$$\text{ii) } \left\{ \begin{array}{l} \frac{x^2}{4} + \frac{y^2}{3} = 1 \\ y = mx + \left(\frac{3}{2} - m \right) \end{array} \right. \rightarrow \left\{ \begin{array}{l} 3x^2 + 4y^2 = 12 \\ y = mx + \left(\frac{3}{2} - m \right) \end{array} \right.$$

$$3x^2 + 4 \left[mx + \left(\frac{3}{2} - m \right) \right]^2 = 12$$



$$3x^2 + 4 \left[mx + \left(\frac{3}{2} - m \right) \right]^2 = 12$$

$$3x^2 + 4 \left[m^2 x^2 + 2mx \left(\frac{3}{2} - m \right) + \left(\frac{3}{2} - m \right)^2 \right] - 12 = 0$$

$$(3 + 4m^2) x^2 + 2m \left(\frac{3}{2} - m \right) \cdot x + \left[\left(\frac{3}{2} - m \right)^2 - 12 \right] = 0$$

$$\rightarrow \Delta = 0 \therefore b^2 = 4ac$$

$$(3m - 2m^2)^2 = 4(3 + 4m^2) \cdot \left[\left(\frac{3}{2} - m \right)^2 - 12 \right]$$

$$9m^2 - 12m^3 + 4m^4 = 4(3 + 4m^2) \left(\frac{9}{4} - 3m + m^2 - 12 \right)$$

$$9m^2 - 12m^3 + 4m^4 = (3 + 4m^2)(4m^2 - 12m - 39)$$

$$9m^2 - 12m^3 + 4m^4 = 12m^2 - 36m - 117 + 16m^4 - 48m^3 - 156m^2$$

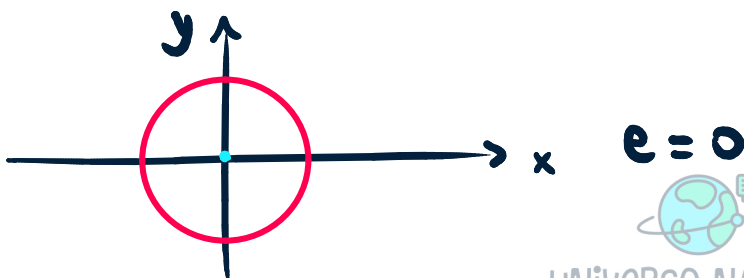
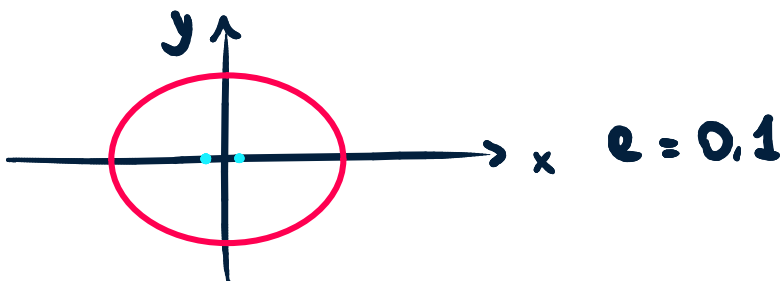
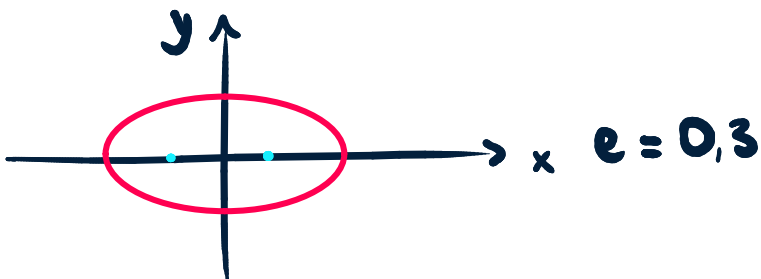
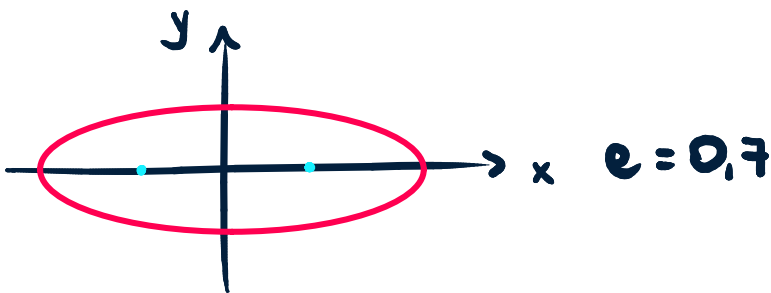
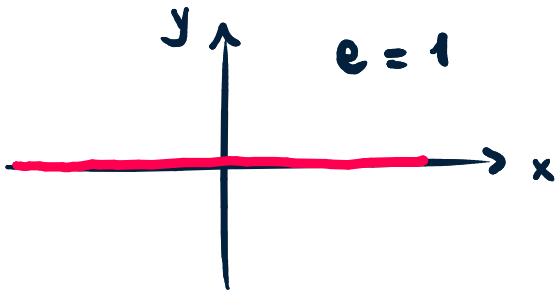
$$12m^4 - 36m^3 - 153m^2 - 36m - 117 = 0$$

→ A SOLUÇÃO DESTA EQUAÇÃO FORNECE O COEF. ANGULAR DA RETA TANGENTE

ELIPSE : EXCENTRICIDADE

$$e := \frac{\text{DISTÂNCIA FOCAL}}{\text{EIXO MAIOR}} = \frac{2c}{2a} = \frac{c}{a}$$

$$0 < e < 1$$



EXEMPLO

Qual é a excentricidade da elipse dada por $50x^2 + 32y^2 = 2$?

R :

$$50x^2 + 32y^2 = 2 \quad \downarrow \div 2$$

$$25x^2 + 16y^2 = 1$$

$$\frac{x^2}{(1/5)^2} + \frac{y^2}{(1/4)^2} = 1$$



$$a^2 = b^2 + c^2$$

$$(1/4)^2 = (1/5)^2 + c^2$$

$$c^2 = \frac{1}{16} - \frac{1}{25}$$

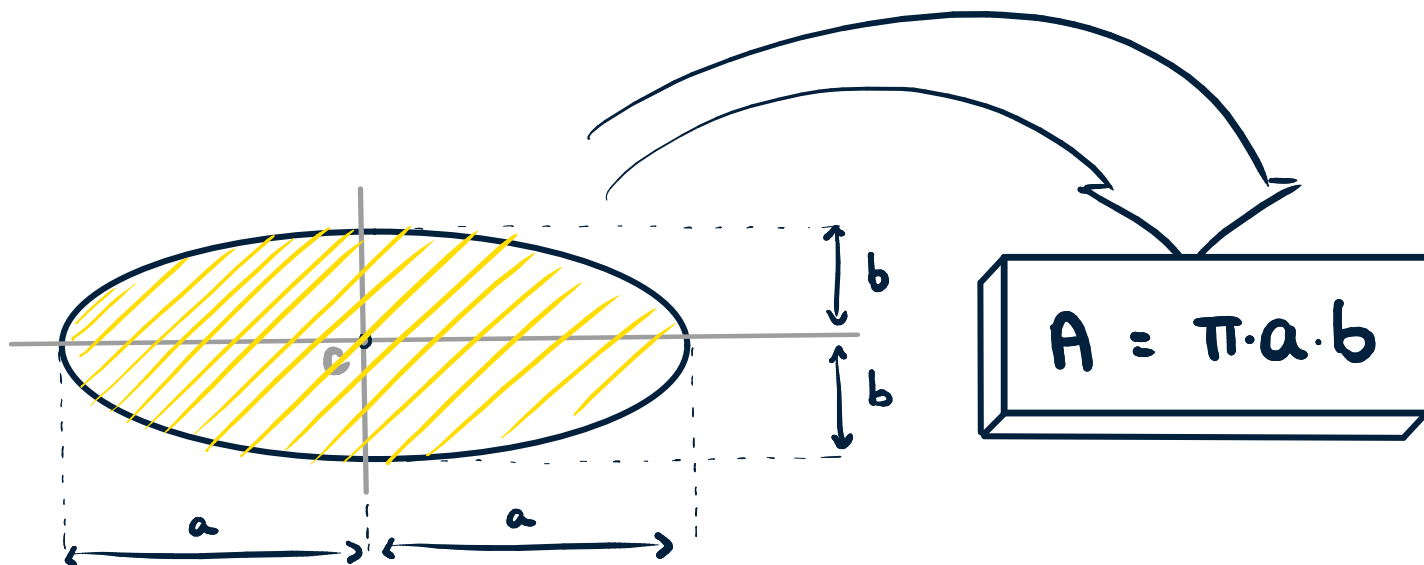
$$c^2 = \frac{25 - 16}{16 \cdot 25}$$

$$c = \frac{3}{4 \cdot 5} = \frac{3}{20}$$

$$e = \frac{c}{a} = \frac{3}{20} \cdot \frac{4}{1} = \frac{3}{5} = 0,6$$

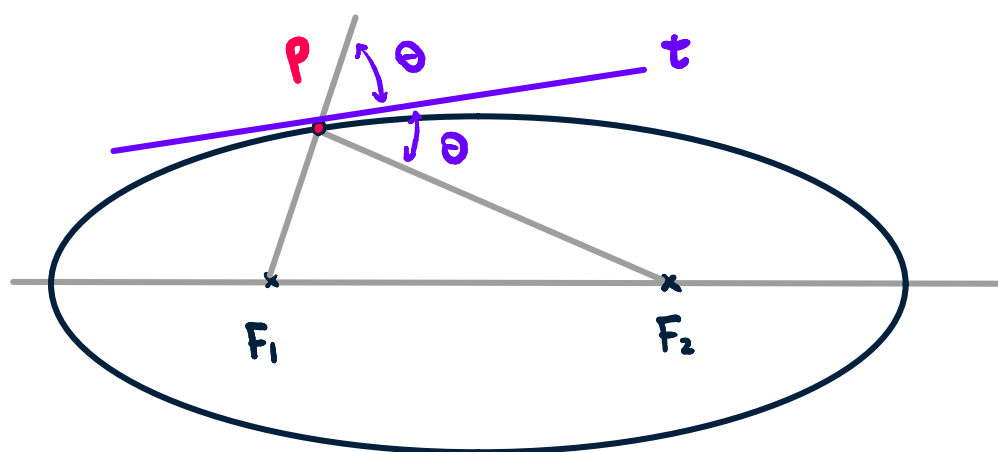
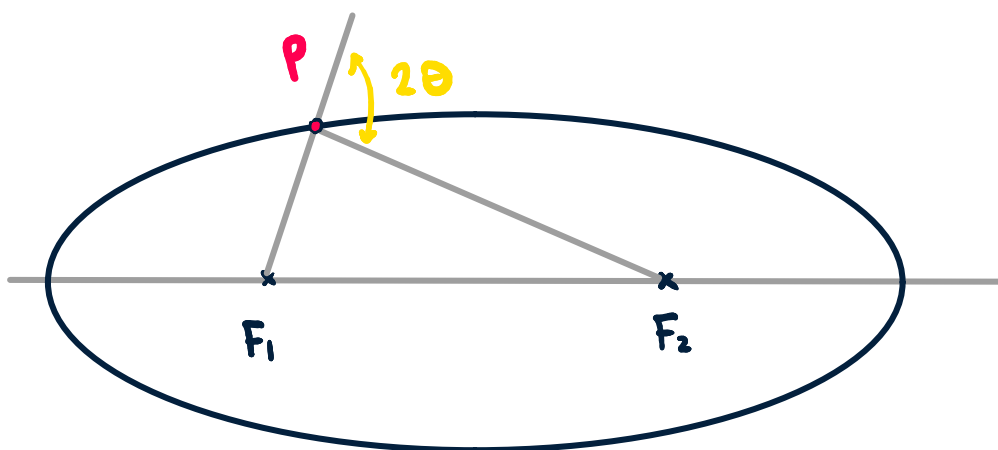
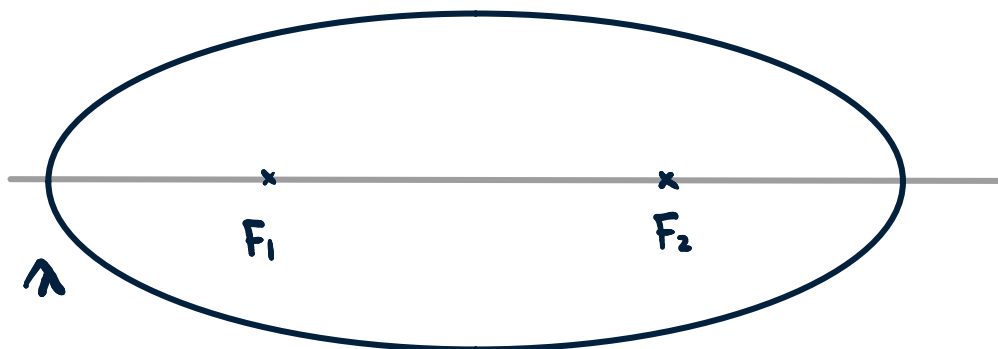


ÁREA DA ELIPSE



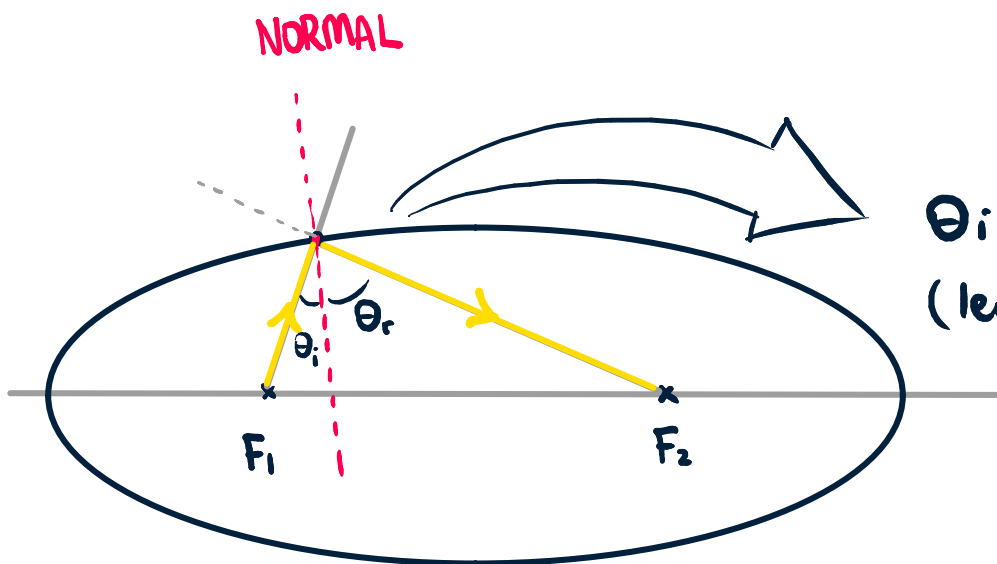
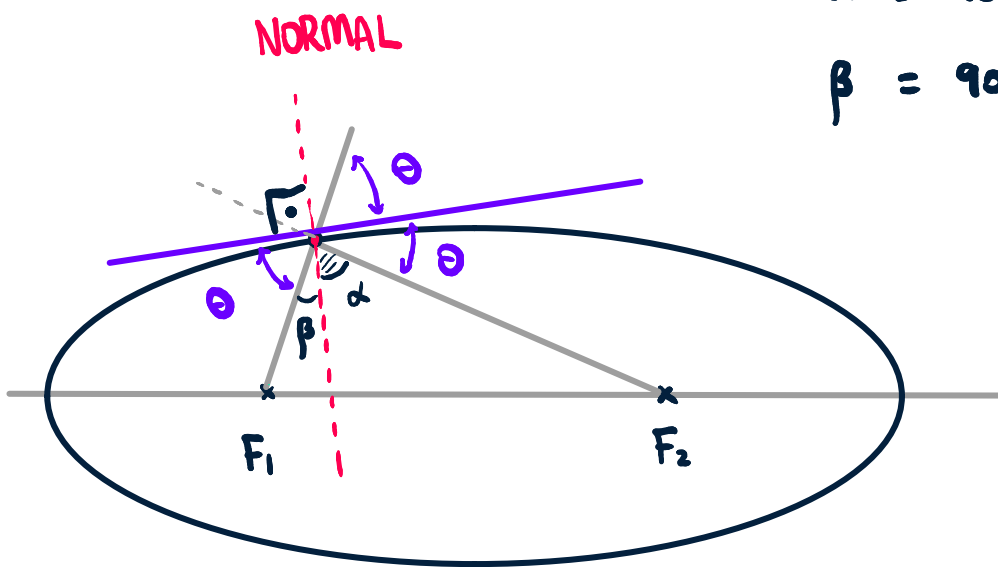
ELIPSE : TEOREMA DAS TANGENTES

Seja elipse λ . Se $P \in \lambda$ e F_1 e F_2 são os focos, então a tangente em P é bissetriz externa de $\widehat{F_1 P F_2}$.



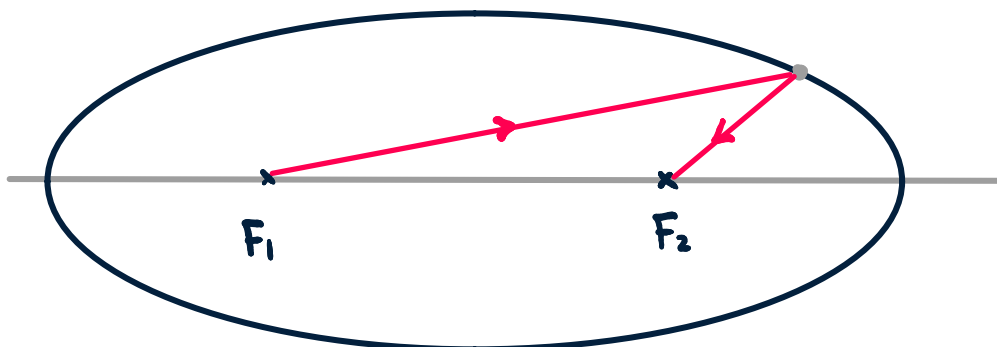
Consequência :

$$\left. \begin{array}{l} \alpha = 90 - \theta \\ \beta = 90 - \theta \end{array} \right\} \alpha = \beta$$



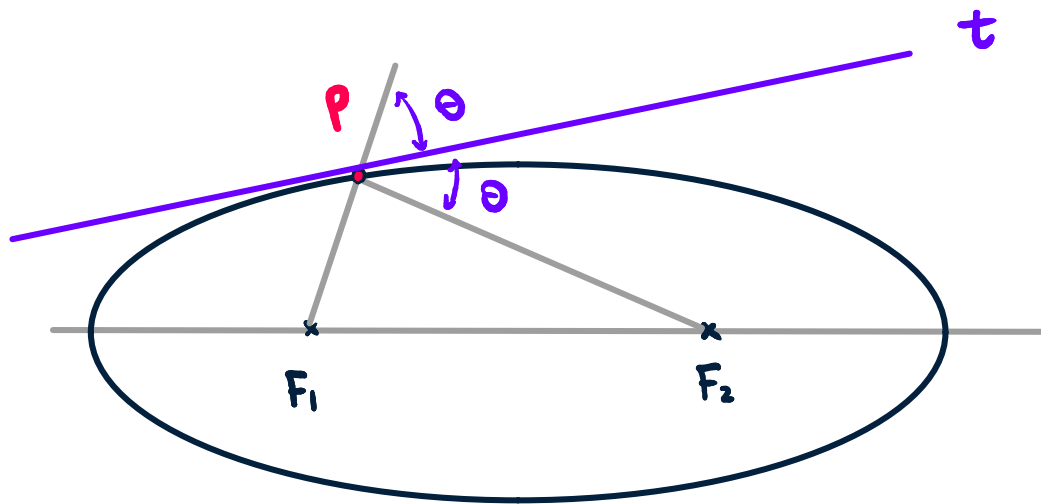
$$\theta_i = \theta_r$$

(lei da reflexão)



Demonstração:

Seja t a bissetriz do ângulo externo $F_1 \hat{P} F_2$:

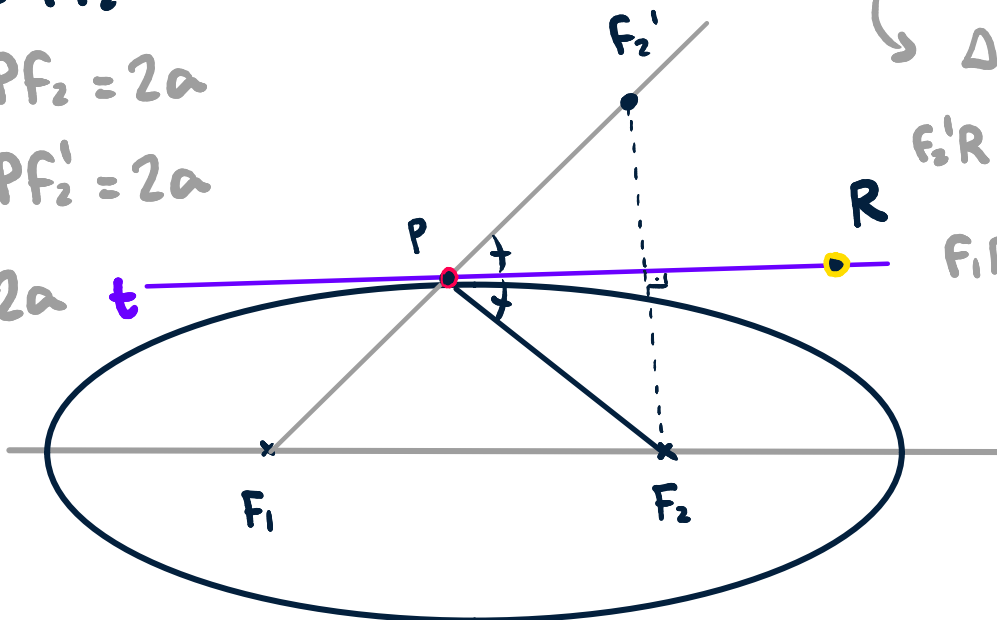


$$PF_2 = PF_2'$$

$$\hookrightarrow PF_1 + PF_2 = 2a$$

$$PF_1 + PF_2' = 2a$$

$$F_1 F_2' = 2a$$



$$F_2' R = F_2 R$$

$$\hookrightarrow \triangle F_1 F_2' R$$

$$F_2' R + F_1 R > F_1 F_2'$$

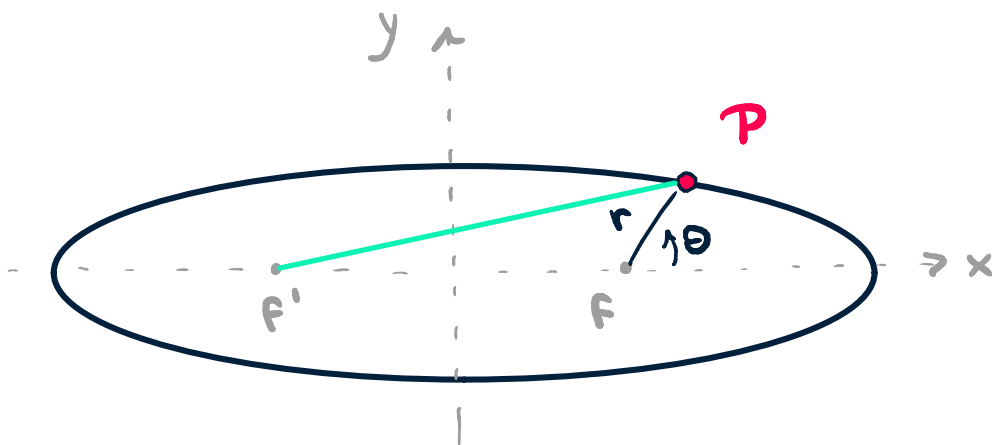
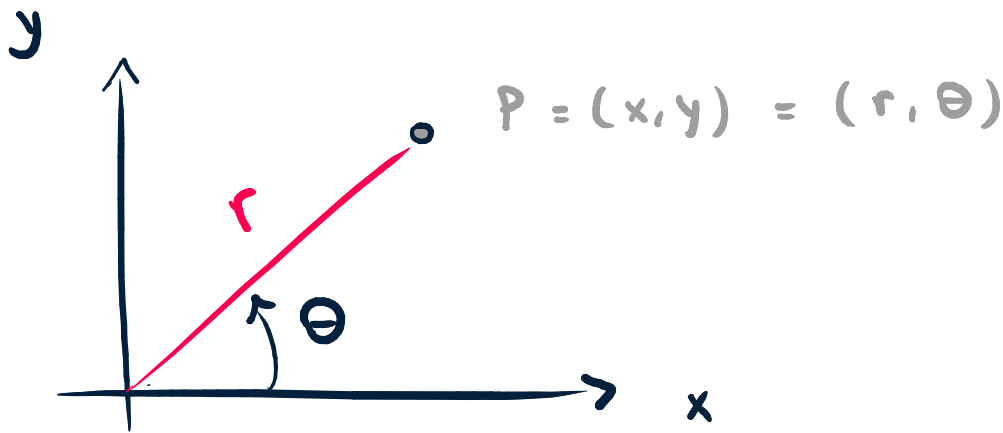
$$F_1 R + F_2 R > 2a$$

□



ELIPSE : EQUAÇÃO POLAR

(origem em um dos focos)



Lei dos Cossenos:

$$(2a - r)^2 = r^2 + 4c^2 - 2 \cdot r \cdot 2c \cdot \underbrace{\cos(\pi - \theta)}_{-\cos \theta}$$

$$4a^2 - 4ar + \cancel{r^2} = \cancel{r^2} + 4c^2 + 2 \cdot r \cdot 2c \cdot \cos \theta$$

$$a^2 - a \cdot r = c^2 + r \cos \theta$$

$$\underbrace{b^2}_{b^2} = a^2 - c^2 = r(a + c \cdot \cos \theta)$$

$$b^2 = r \cdot a \left(1 + \frac{c}{a} \cos \theta\right)$$



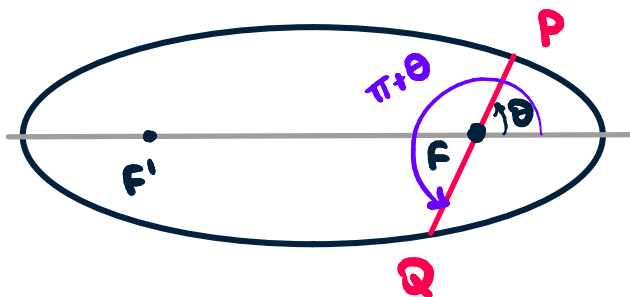
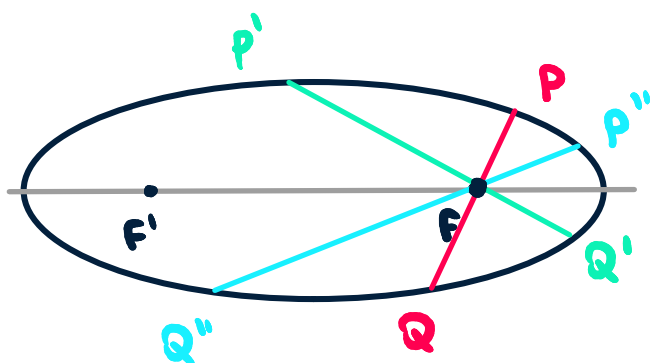
$$r(\theta) = \frac{b^2}{a(1 + e \cdot \cos \theta)}$$

EXEMPLO

Seja uma elipse λ e $P, Q \in \lambda$ tal que $\overline{PQ}$ é uma corda focal, isto é, contém o foco.

Qual é o menor valor de $\overline{PQ}$?

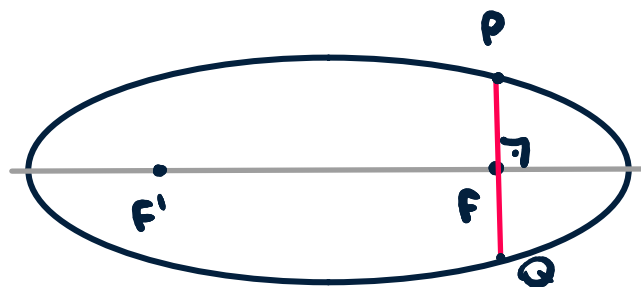
R:



$$r(\theta) = \frac{b^2}{a(1 + e \cdot \cos \theta)}$$
$$\begin{aligned} \rightarrow FP &= \frac{b^2}{a(1 + e \cdot \cos \theta)} \\ \hookrightarrow FQ &= \frac{b^2}{a[1 + e \cdot \cos(\pi + \theta)]} \end{aligned}$$

$$PQ = PF + FQ = \frac{b^2}{a} \left[\frac{1}{1 + e \cdot \cos \theta} + \frac{1}{1 - e \cdot \cos \theta} \right]$$

$$PQ = \frac{b^2}{a} \left[\frac{1 - \cancel{e \cdot \cos \theta} + 1 + \cancel{e \cdot \cos \theta}}{(1 + e \cdot \cos \theta)(1 - e \cdot \cos \theta)} \right]$$



$$PQ(\theta) = \frac{2b^2}{a} \cdot \frac{1}{(1 - e^2 \cos^2 \theta)}$$



$$PQ_{\min} = \frac{2b^2}{a} \quad (\theta = 90^\circ)$$

ELIPSE : EQUAÇÃO PARAMÉTRICA

$$\frac{(x - x_c)^2}{a^2} + \frac{(y - y_c)^2}{b^2} = 1$$

↳ $(x - x_c) = a \cdot \sin(t) \therefore x(t) = x_c + a \cdot \sin(t)$

↳ $(y - y_c) = b \cdot \cos(t) \therefore y(t) = y_c + b \cdot \cos(t)$



EXEMPLO

Um corpo se move segundo as equações

$$x(t) = 2 \cos t$$

$$y(t) = 5 \sin t.$$

Determine o instante t no qual o corpo está o mais afastado possível da origem.

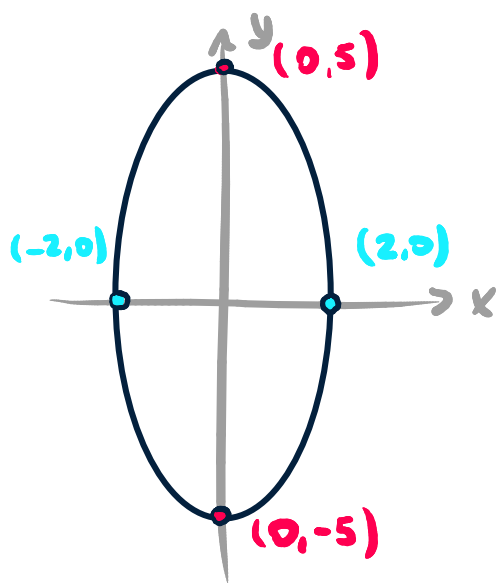
R :

$$\frac{x(t)}{2} = \cos t$$

$$\frac{y(t)}{5} = \sin t$$

$$\sin^2 t + \cos^2 t = 1$$

$$\frac{x^2}{2^2} + \frac{y^2}{5^2} = 1$$



$$y(t) = \pm 5$$

$$5 \sin t = \pm 5$$

$$\sin t = \pm 1$$

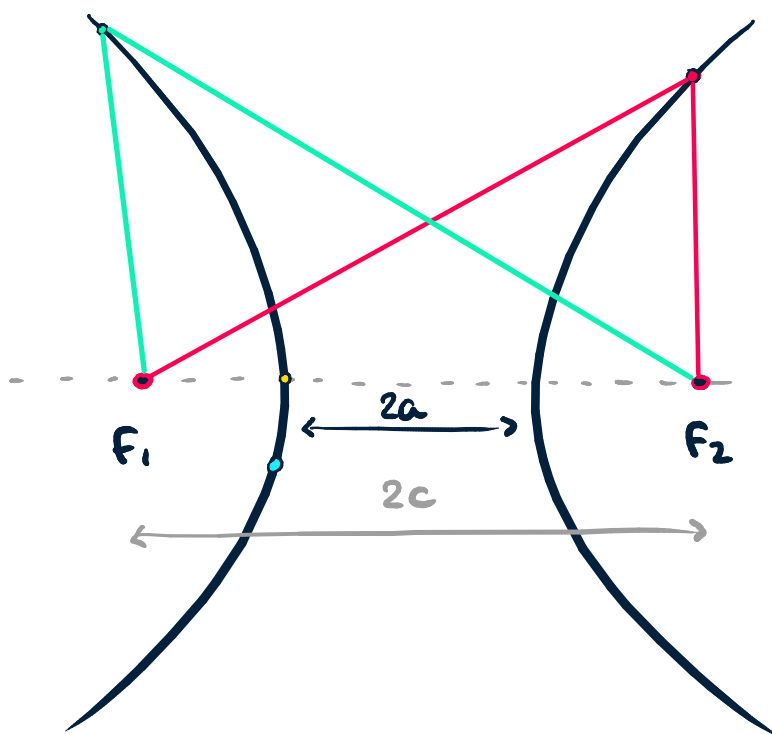
$$t = \frac{\pi}{2} + k \cdot \pi, k \in \mathbb{Z}$$

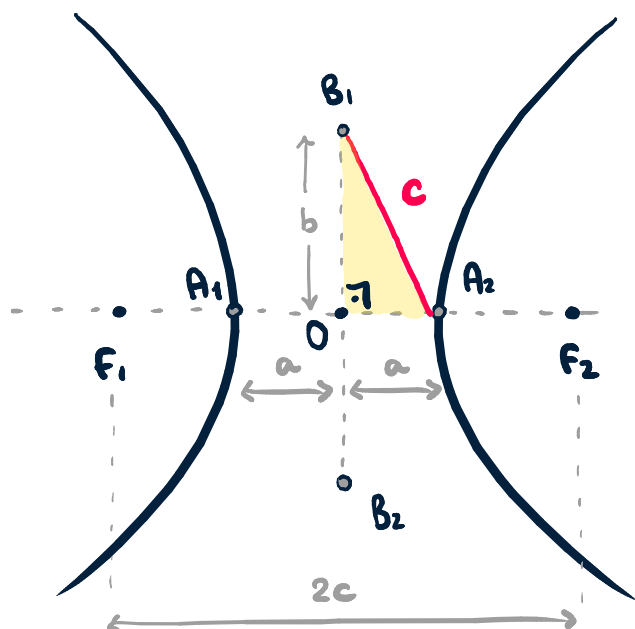




DEFINIÇÃO

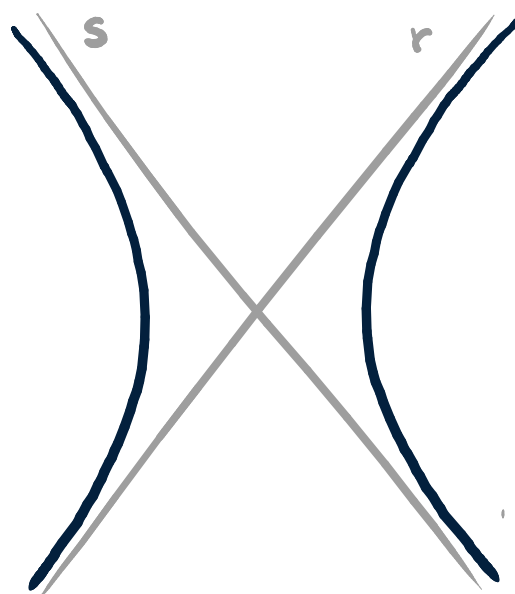
Uma hipérbole é o conjunto dos pontos P tal que $|\overline{PF_1} - \overline{PF_2}| = 2a = \text{constante}$, onde F_1 e F_2 são seus focos e $\overline{F_1F_2} = 2c$ é a distância focal.





RELAÇÃO NOTÁVEL

$$c^2 = a^2 + b^2$$



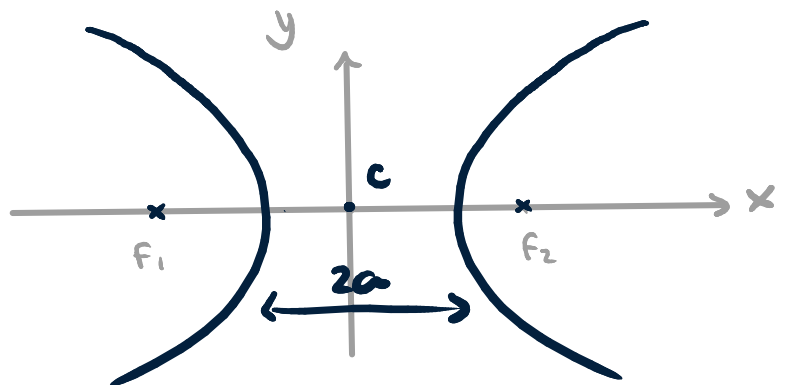
- F_1 e F_2 : focos da hipérbole
- O : centro
- $\overline{A_1 A_2} = 2a$: eixo real ou transverso
- $\overline{B_1 B_2} = 2b$: eixo imaginário
- $\overline{F_1 F_2} = 2c$: distância focal
- r e s : assintotas

HIPÉRBOLE: EQUAÇÃO REDUZIDA

A equação reduzida da hipérbole de eixo real $2a$ e eixo imaginário $2b$, paralelos aos eixos x e y , respectivamente, é:

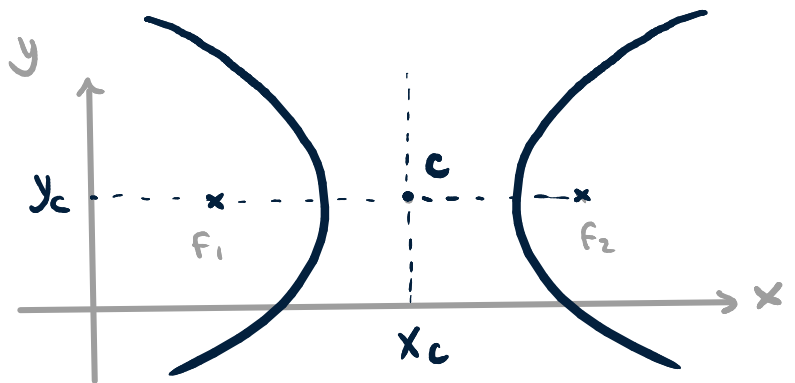
$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

centro na origem



$$\frac{(x-x_c)^2}{a^2} - \frac{(y-y_c)^2}{b^2} = 1$$

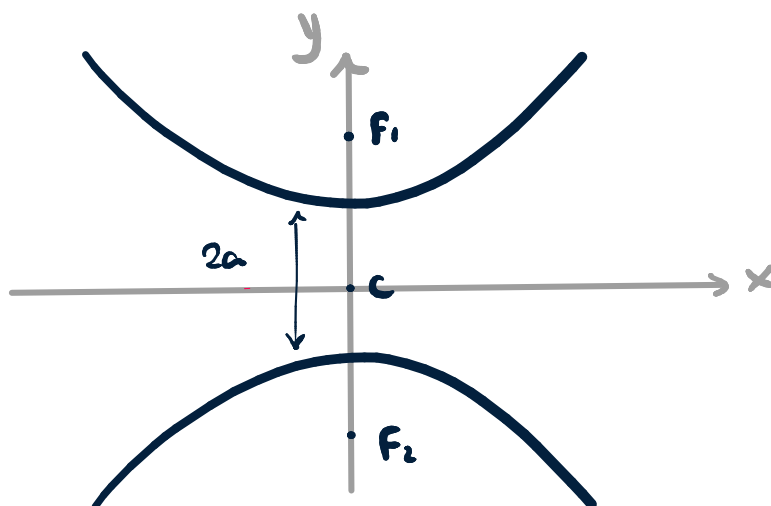
centro em (x_c, y_c)



Se o eixo real for vertical:

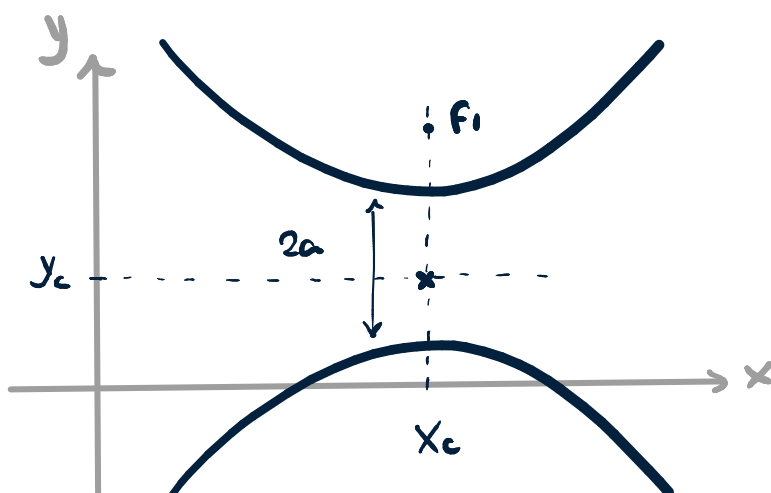
$$-\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$$

centro na origem



$$-\frac{(x-x_c)^2}{b^2} + \frac{(y-y_c)^2}{a^2} = 1$$

centro em (x_c, y_c)



EXEMPLO

Encontre as coordenadas dos focos da hipérbole

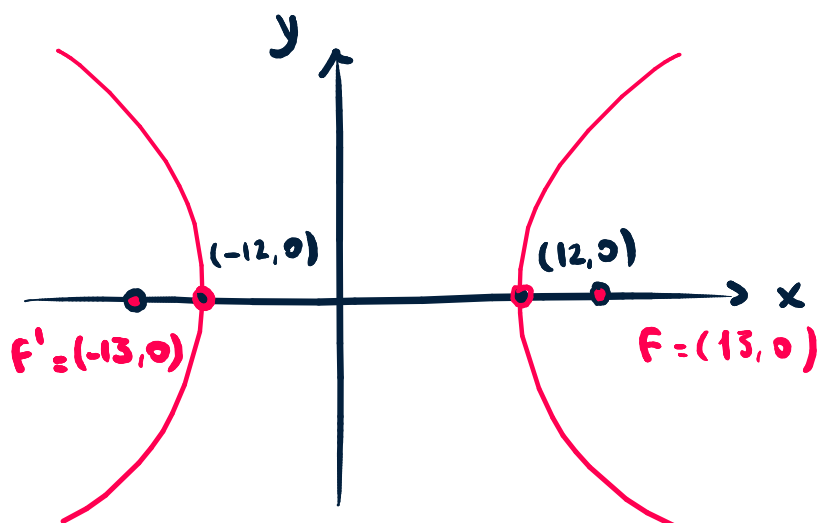
$$\frac{x^2}{12^2} - \frac{y^2}{5^2} = 1$$

R :

$$\frac{x^2}{12^2} - \frac{y^2}{5^2} = 1$$

$a = 12$
 $b = 5$

$$c^2 = a^2 + b^2 = 12^2 + 5^2 = 169 \quad \therefore \boxed{c = 13}$$

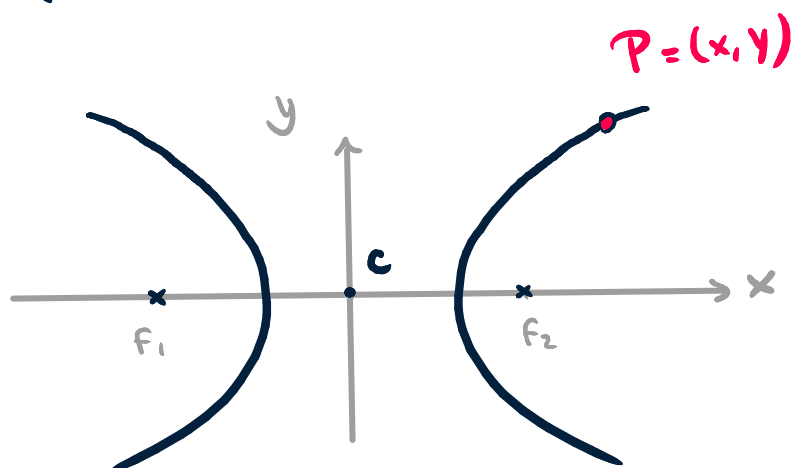


R : $F_1 = (13, 0)$ e $F_2 = (-13, 0)$



HIPÉRBOLE: EQUAÇÃO REDUZIDA

(DEMONSTRAÇÃO)



$$F_1 = (-c, 0)$$

$$F_2 = (c, 0)$$

$$|d(P; F_1) - d(P; F_2)| = 2a$$

$$|\sqrt{(x+c)^2 + (y-0)^2} - \sqrt{(x-c)^2 + (y-0)^2}| = 2a$$

$$\sqrt{(x+c)^2 + y^2} - \sqrt{(x-c)^2 + y^2} = 2a$$

$$\left(\sqrt{(x+c)^2 + y^2} \right)^2 = \left(2a + \sqrt{(x-c)^2 + y^2} \right)^2$$

$$\cancel{x^2} + 2xc + \cancel{c^2} + \cancel{y^2} = 4a^2 + \cancel{x^2} - 2xc + \cancel{c^2} + \cancel{y^2} + 4a\sqrt{(x-c)^2 + y^2}$$

$$4xc = 4a^2 + 4a\sqrt{(x-c)^2 + y^2}$$

$$xc - a^2 = a\sqrt{(x-c)^2 + y^2}$$



$$(xc - a^2)^2 = (a\sqrt{(x-c)^2 + y^2})^2$$

$$x^2c^2 - 2xca^2 + a^4 = a^2(x^2 - 2xc + c^2 + y^2)$$

$$x^2c^2 - 2xca^2 + a^4 = a^2x^2 - 2xca^2 + a^2c^2 + a^2y^2$$

$$x^2(c^2 - a^2) - a^2y^2 = a^2c^2 - a^4 = a^2(c^2 - a^2)$$

$$b^2x^2 - a^2y^2 = a^2 \cdot b^2$$

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

□

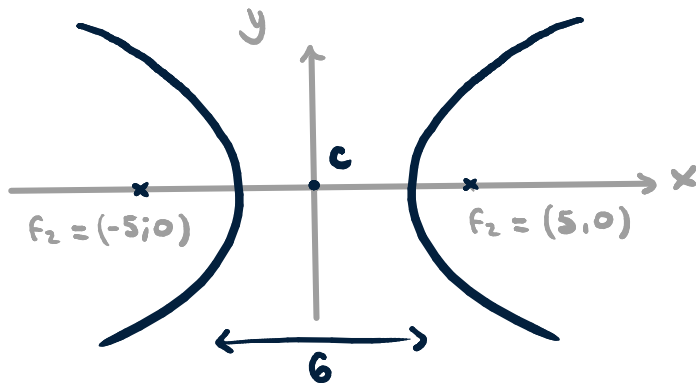
↓ $\div a^2b^2$



EXEMPLO 01

Escreva equação da hipérbole:

(i)

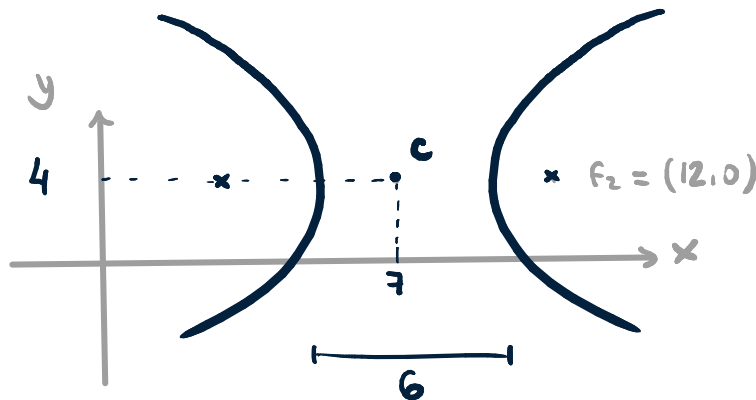


$$\begin{aligned} 2a &= 6 \therefore a = 3 \\ 2c &= 10 \therefore c = 5 \\ b &= 4 \text{ (PITÁGORAS)} \end{aligned}$$

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

$$\boxed{\frac{x^2}{9} - \frac{y^2}{16} = 1}$$

(ii)

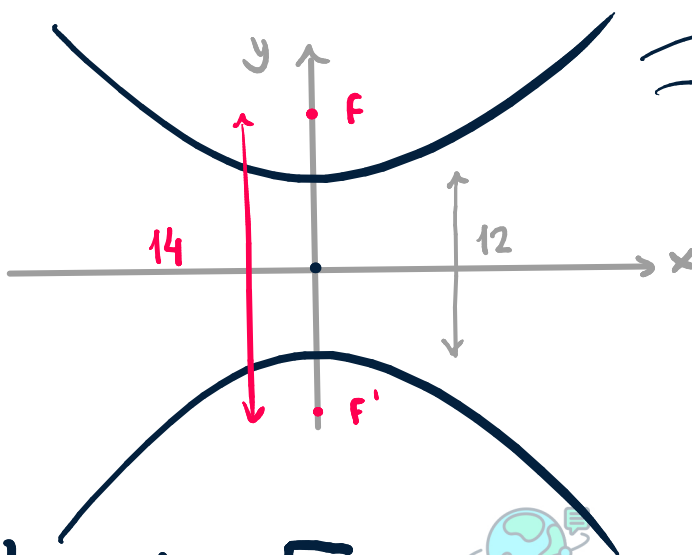


$$\frac{(x-x_c)^2}{a^2} - \frac{(y-y_c)^2}{b^2} = 1$$

$$\boxed{\frac{(x-7)^2}{9} - \frac{(y-4)^2}{16} = 1}$$

$$\begin{aligned} 2a &= 6 \therefore a = 3 \\ 2c &= 10 \therefore c = 5 \end{aligned} \left. \vphantom{\begin{aligned} 2a &= 6 \\ 2c &= 10 \end{aligned}} \right\} \begin{aligned} c^2 &= a^2 + b^2 \\ 5^2 &= 3^2 + b^2 \therefore b = 4 \end{aligned}$$

(iii)



$$a = 6$$

$$c = 7$$

$$c^2 = a^2 + b^2 \rightarrow b = \sqrt{13}$$

$$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$$

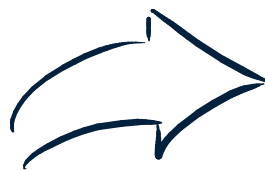
$$\boxed{\frac{y^2}{36} - \frac{x^2}{13} = 1}$$

EXEMPLO 02

Determine as coordenadas dos focos da hipérbole $\frac{(x-3)^2}{5} - \frac{(y-2)^2}{7} = 1$.

R :

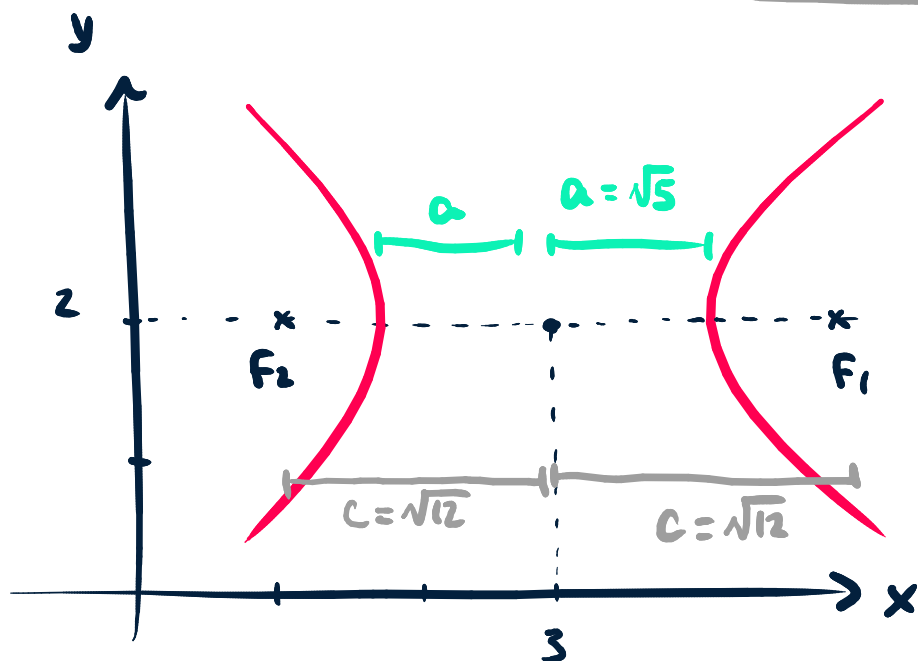
$$\frac{(x-3)^2}{5} - \frac{(y-2)^2}{7} = 1$$



$$C = (3, 2)$$

$$a = \sqrt{5} \quad b = \sqrt{7}$$

$$c^2 = a^2 + b^2 = 12 \therefore \boxed{c = \sqrt{12}}$$



$$f_1 = (3 + \sqrt{12}; 2)$$

$$f_2 = (3 - \sqrt{12}; 2)$$



EXEMPLO 03 (ITA 2006)

Calcule a distância focal da hipérbole

$$9y^2 - 16x^2 - 144y - 224x - 352 = 0.$$

R :

$$9y^2 - 144y - 16x^2 - 224x - 352 = 0$$

$$9(y^2 - 16y) - 16(x^2 + 14x) - 352 = 0$$

$$9(y^2 - 16y + \square) - 16(x^2 + 14x + \bigcirc) - 352 = 9 \cdot \square - 16 \cdot \bigcirc$$

$$9(y^2 - 16y + 8^2) - 16(x^2 + 14x + 7^2) - 352 = 9 \cdot 8^2 - 16 \cdot 7^2$$

$$9 \cdot (y - 8)^2 - 16 \cdot (x + 7)^2 = 352 + 64 \cdot 9 - 16 \cdot 49$$

$$9 \cdot (y - 8)^2 - 16 \cdot (x + 7)^2 = 144 \quad \left. \begin{array}{l} \end{array} \right\} \div 144$$

$$\frac{(y - 8)^2}{4^2} - \frac{(x + 7)^2}{3^2} = 1$$

$$\begin{array}{l} a = 4 \\ b = 3 \end{array}$$

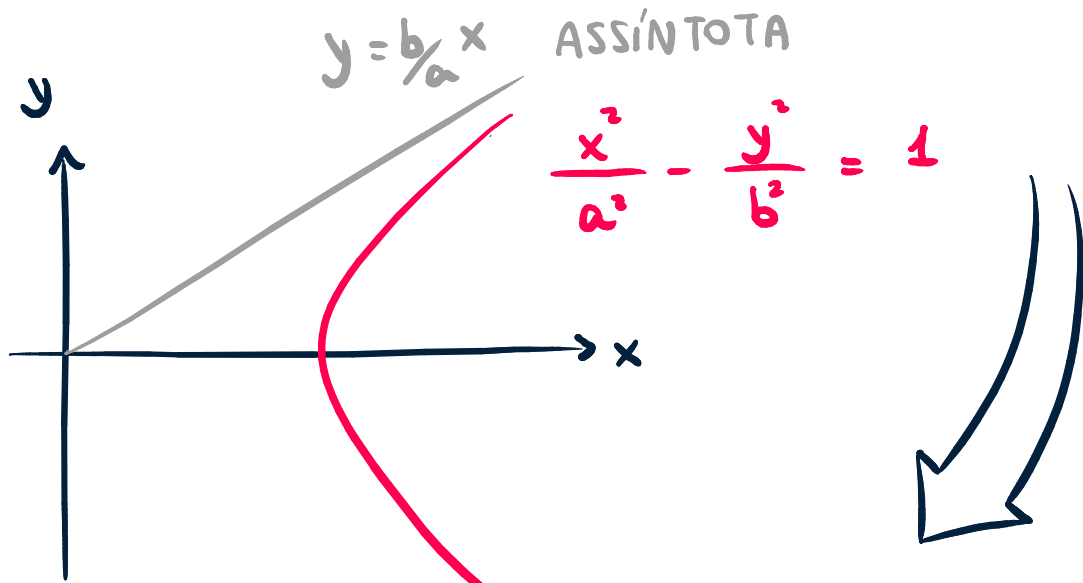
$$\left. \begin{array}{l} a = 4 \\ b = 3 \end{array} \right\} c^2 = a^2 + b^2 \therefore$$

$$c = 5$$

$$2c = 10$$

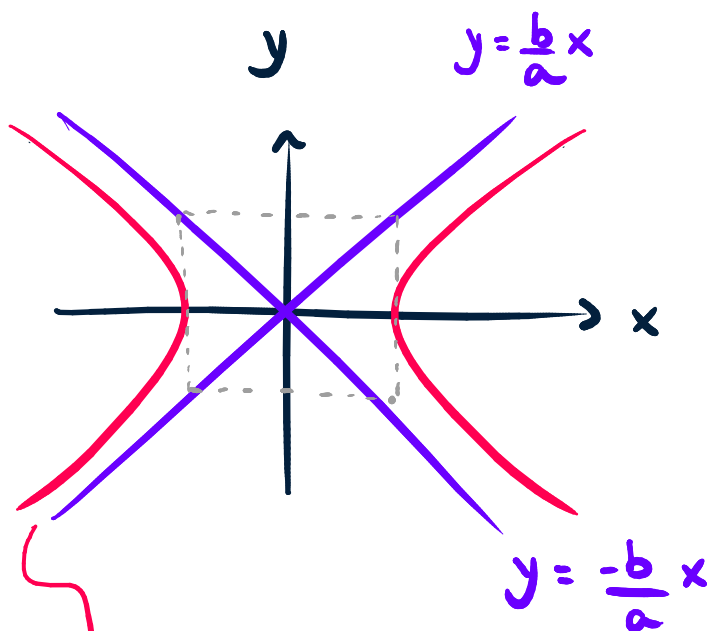


HIPÉRBOLE: EXCENTRICIDADE e ASSÍNTOTAS

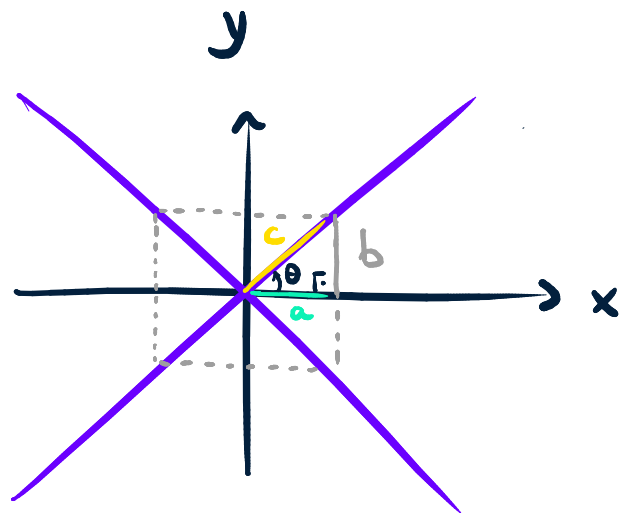


$$y^2 = \frac{b^2}{a^2}x^2 - b^2 \frac{x^2}{a^2} \cdot \frac{a^2}{x^2} \therefore y^2 = \left(\frac{b}{a}x\right)^2 \left[1 - \frac{a^2}{x^2}\right]$$

$$y = \pm \frac{b}{a}x \sqrt{1 - \frac{a^2}{x^2}} \xrightarrow{x \rightarrow \pm \infty} \boxed{y = \pm \frac{b}{a}x}$$

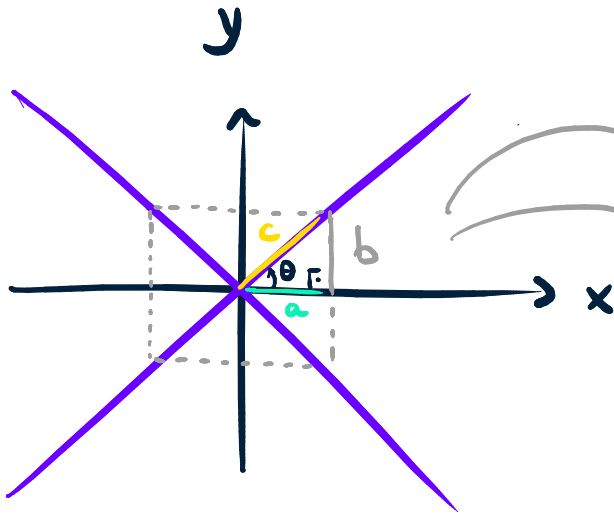


$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

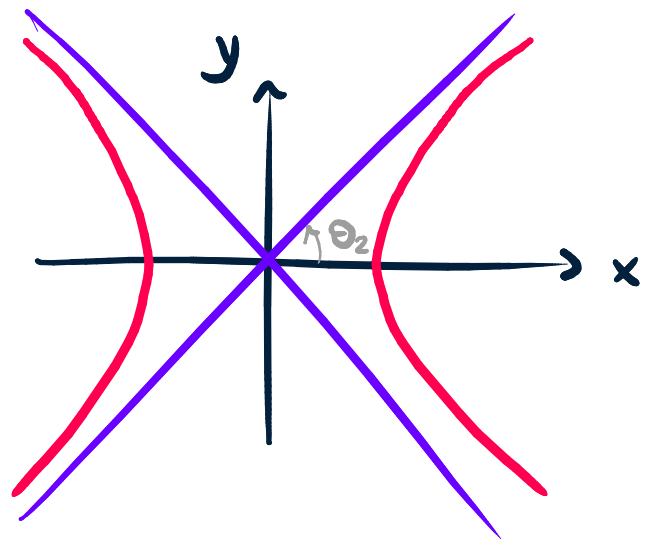
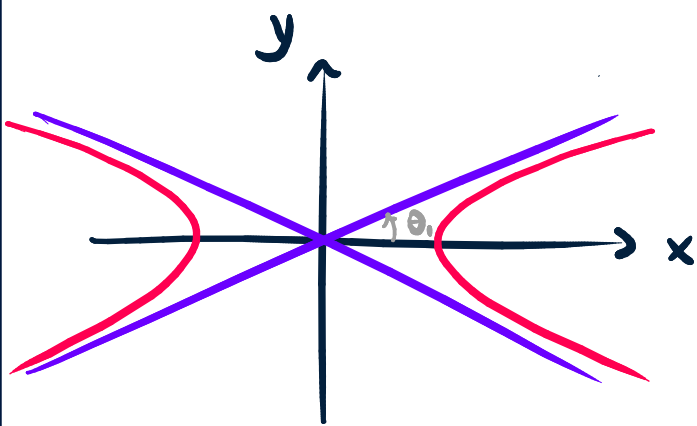


A excentricidade:

$$e := \frac{c}{a}$$



$$\cos \Theta = \frac{a}{c} = \frac{1}{e}$$



$$\Theta_1 < \Theta_2$$

$$e_1 > e_2$$



EXEMPLO

Qual é a equação da hipérbole de distância focal 6 e assintotas $y = \pm 2x$?

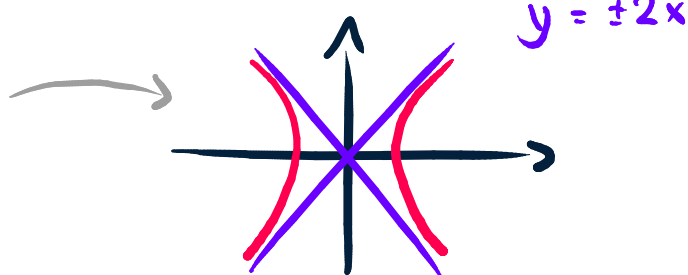
R : $y = \pm \frac{b}{a}x \quad \therefore \quad \frac{b}{a} = 2 \quad \therefore \quad \boxed{b = 2a}$

$\hookrightarrow 2c = 6 \quad \therefore \quad \boxed{c = 3}$

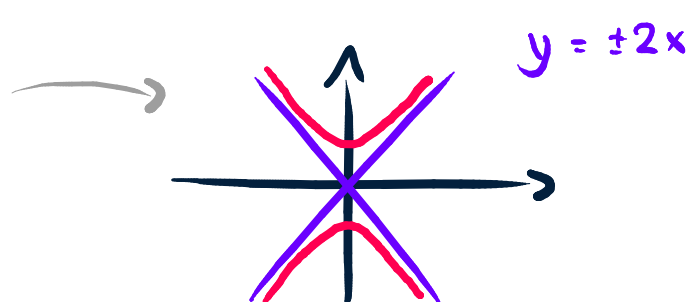
$\rightarrow c^2 = a^2 + b^2 \quad \therefore \quad 3^2 = a^2 + (2a)^2$

$\boxed{a^2 = \frac{9}{5}} \quad \text{e} \quad b = 2a \quad \therefore \quad \boxed{b^2 = 4a^2 = \frac{36}{5}}$

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$



$$-\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$$



HIPÉRBOLE: EQUAÇÃO PARAMÉTRICA

$$\frac{(x-x_c)^2}{a^2} - \frac{(y-y_c)^2}{b^2} = 1$$

$$\sec^2 \theta - \tan^2 \theta = 1$$

Seja $(x-x_c) = a \cdot \sec \theta \quad \therefore$

$(y-y_c) = b \cdot \tan \theta \quad \therefore$

$$x(\theta) = x_c + a \cdot \sec \theta$$

$$y(\theta) = y_c + b \cdot \tan \theta$$



EXEMPLO

Encontre a distância focal da cônica descrita por:

$$\begin{cases} x = 3 \cdot \sec(t) \\ y = 8 \cdot \tan(t) \end{cases}$$

R :

$$\begin{cases} x = 3 \cdot \sec(t) \\ y = 8 \cdot \tan(t) \end{cases} \quad \therefore \quad \begin{aligned} x/3 &= \sec(t) \\ y/8 &= \tan(t) \end{aligned}$$

$$\hookrightarrow 1 + \tan^2 t = \sec^2 t \quad \therefore \quad 1 + \frac{y^2}{8^2} = \frac{x^2}{3^2}$$

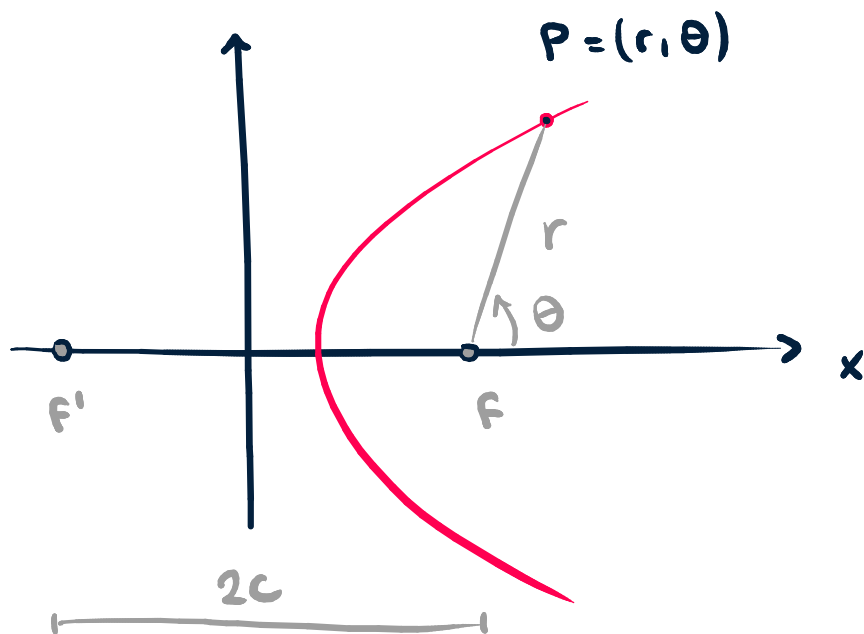
$$\frac{x^2}{3^2} - \frac{y^2}{8^2} = 1 \quad \begin{aligned} &\rightarrow a = 3 \\ &\rightarrow b = 8 \\ &\hookrightarrow c^2 = b^2 + a^2 = 8^2 + 3^2 = 73 \end{aligned}$$

$$\boxed{2c = 2\sqrt{73}}$$



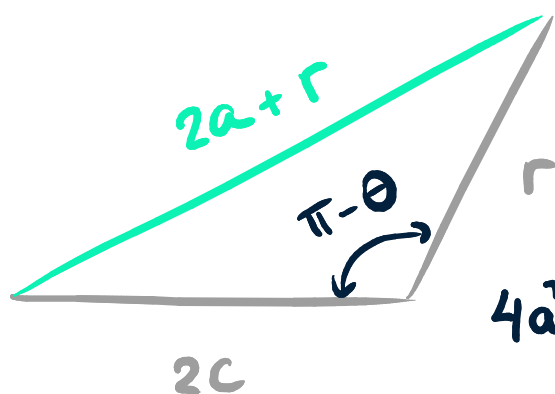
HIPÉRBOLE: EQUAÇÃO POLAR

(origem em um dos focos)



Lei dos Cossenos :

$$(2a+r)^2 = r^2 + 4c^2 - 2 \cdot r \cdot 2c \cdot \underbrace{\cos(\pi-\theta)}_{-\cos\theta}$$



$$4a^2 + \cancel{r^2} + 4ar = \cancel{r^2} + 4c^2 + 2 \cdot r \cdot 2c \cdot \cos\theta$$

$$a^2 + ar = c^2 + r \cdot c \cdot \cos\theta$$

$$r(a - c \cdot \cos\theta) = c^2 - a^2 = b^2 \therefore r = \frac{b^2}{a(1 - e \cos\theta)}$$

$$r(\theta) = \frac{b^2}{a(1 - e \cos\theta)}$$



EXEMPLO

Determine a equação reduzida da hipérbole

$$r(\theta) = \frac{12}{3 - 4\cos\theta}.$$

R :

$$r(\theta) = \frac{12}{3 - 4\cos\theta} = \frac{b^2}{a - a \cdot e \cdot \cos\theta}$$

$$\left. \begin{array}{l} \text{(i)} \quad b^2 = 12 \\ \text{(ii)} \quad a = 3 \end{array} \right\} \quad \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

$$\boxed{\frac{x^2}{9} - \frac{y^2}{12} = 1}$$

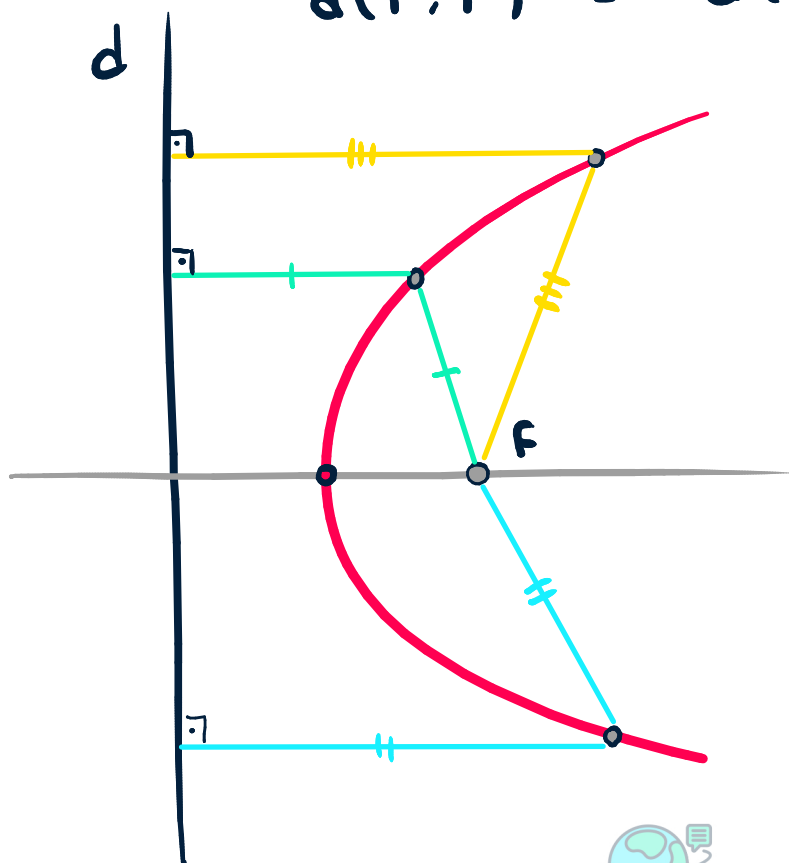


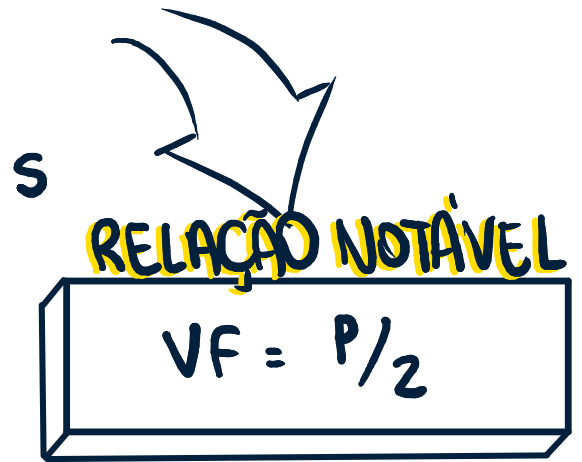
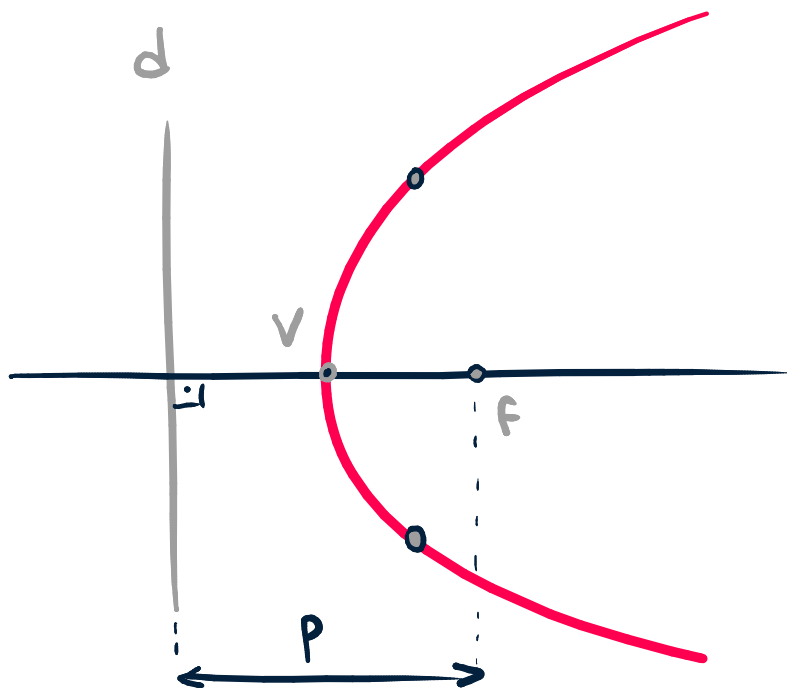


DEFINIÇÃO

Dados um ponto F (foco) e uma reta d (diretriz), uma parábola é o conjunto dos pontos P que estão a uma mesma distância de F e de d :

$$d(P, F) = d(P, d)$$



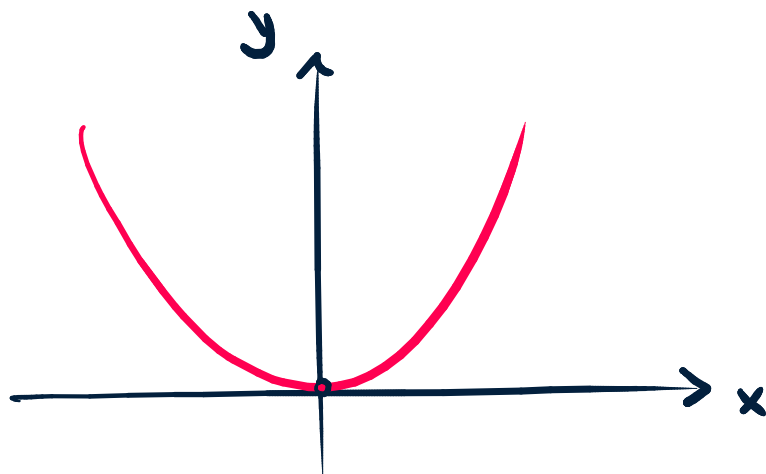


- v : vértice
- f : foco
- d : reta diretriz
- p : parâmetro
- s : eixo de simetria

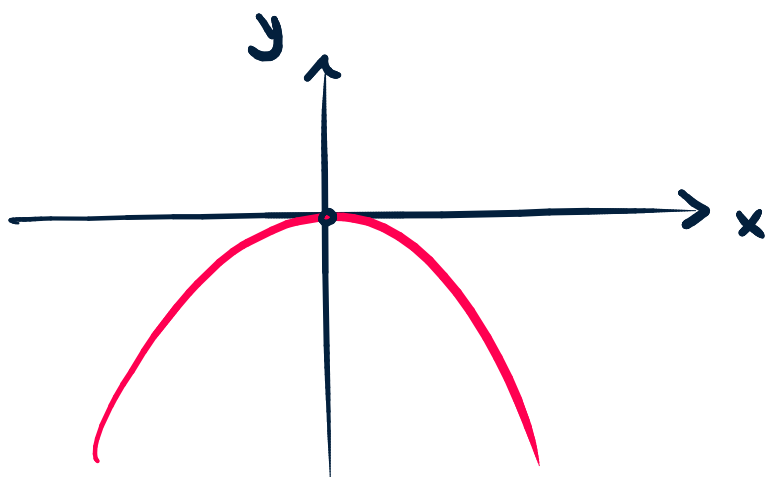


PARÁBOLA: EQUAÇÃO REDUZIDA

EIXO DE SIMETRIA VERTICAL

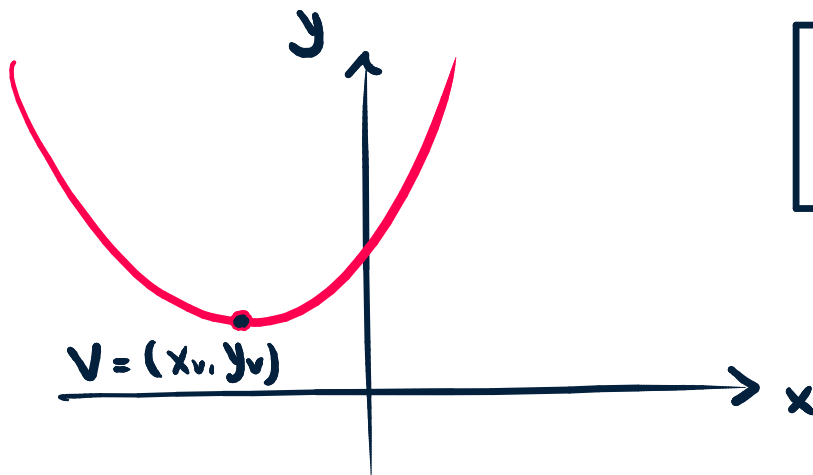


$$x^2 = 2p \cdot y$$

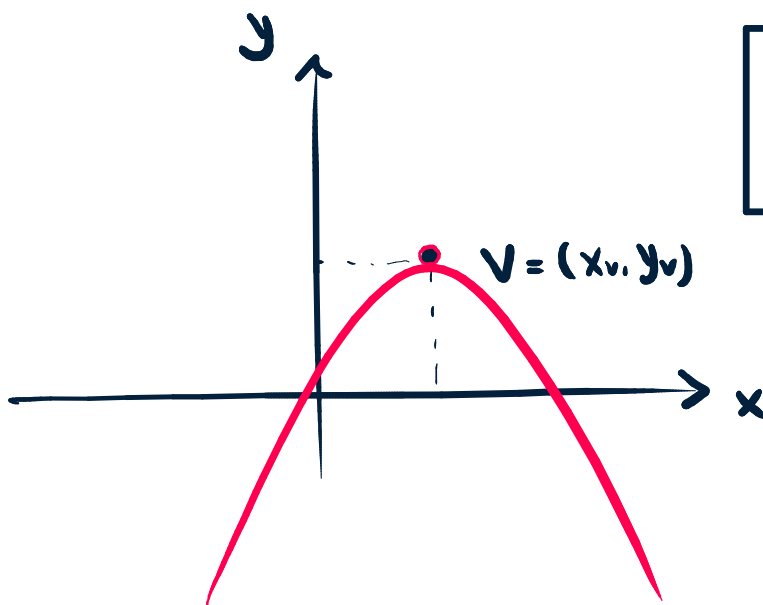


$$x^2 = -2p \cdot y$$





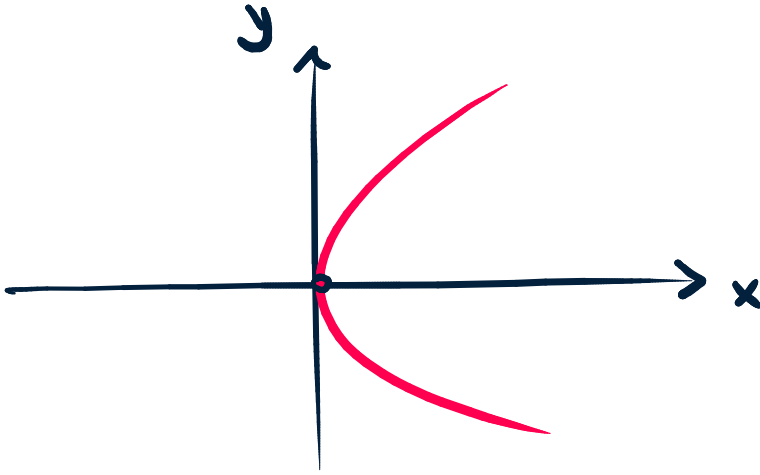
$$(x - x_v)^2 = 2p(y - y_v)$$



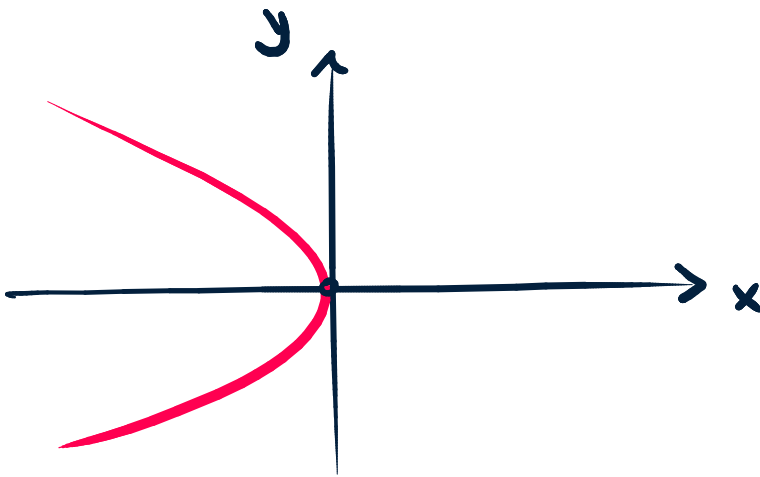
$$(x - x_v)^2 = -2p(y - y_v)$$



EIXO DE SIMETRIA HORIZONTAL

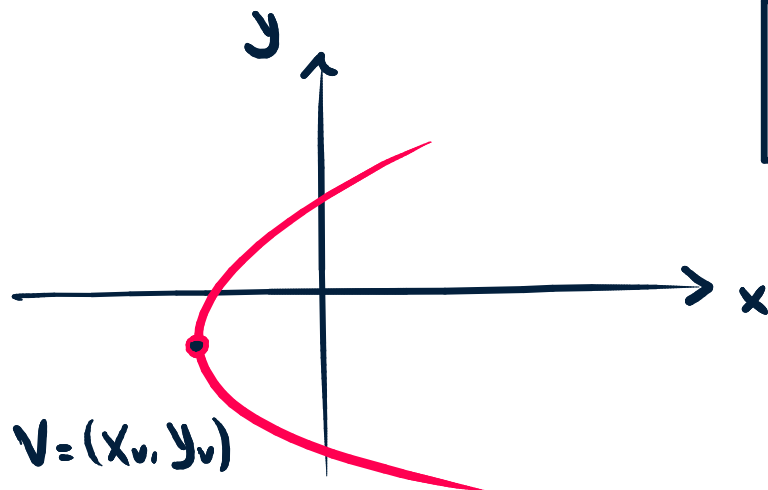


$$y^2 = 2p \cdot x$$

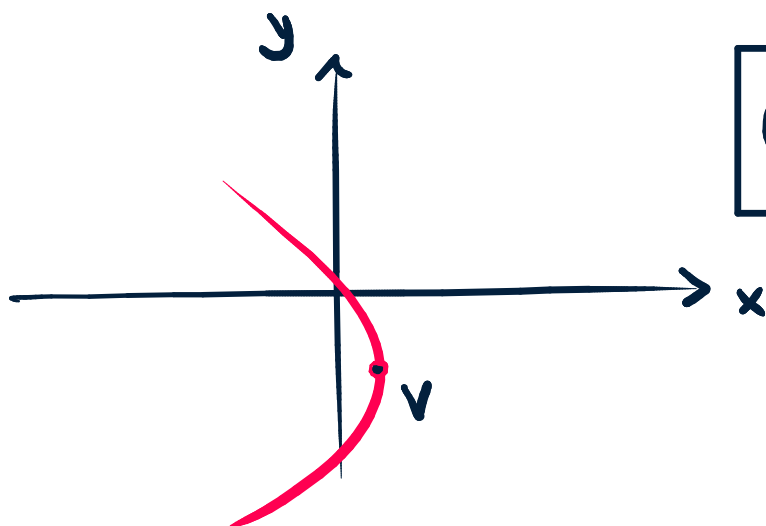


$$y^2 = -2p \cdot x$$





$$(y - y_v)^2 = 2p(x - x_v)$$

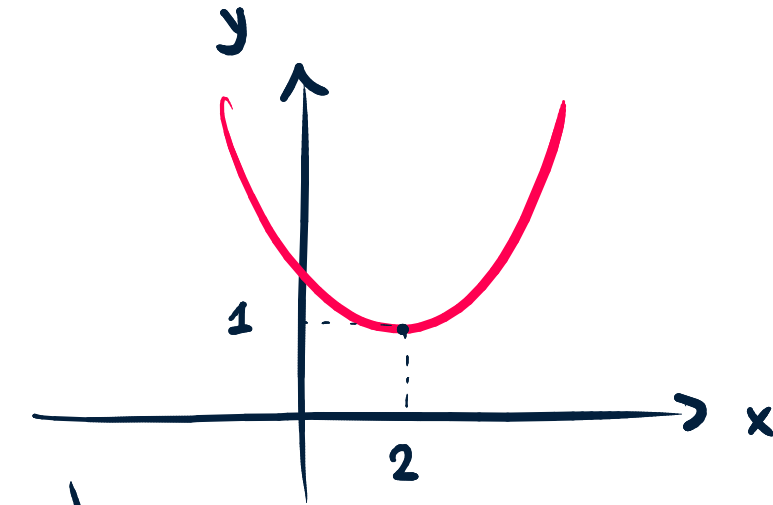


$$(y - y_v)^2 = -2p(x - x_v)$$



EXEMPLO

$$p = 3$$



$$(x - x_v)^2 = 2p \cdot (y - y_v)$$

$$(x - 2)^2 = 2 \cdot 3 \cdot (y - 1)$$

$$(x - 2)^2 = 6 \cdot (y - 1)$$

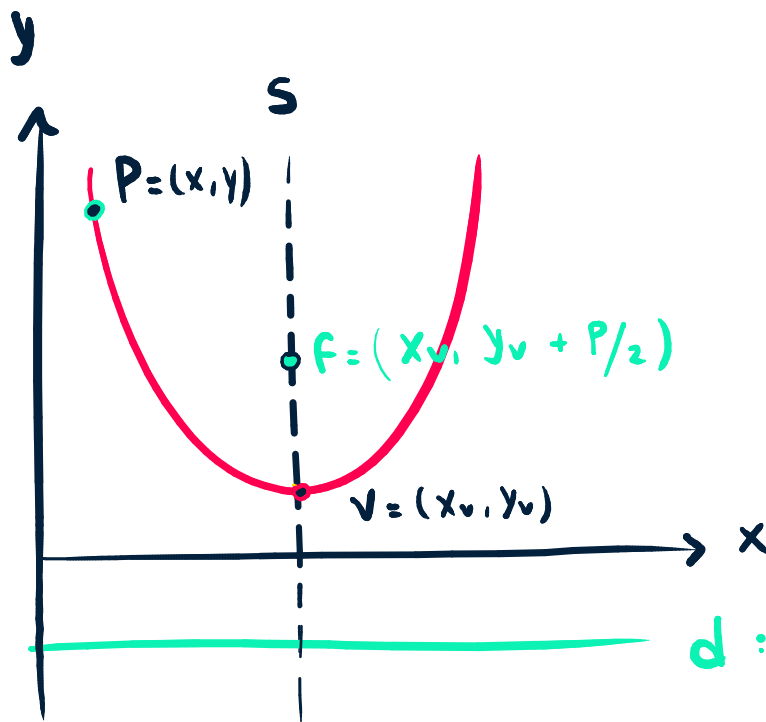
$$x^2 - 4x + 4 = 6y - 6$$

$$y(x) = \frac{1}{6} (x^2 - 4x + 10)$$



PARÁBOLA: EQUAÇÃO REDUZIDA

DEMONSTRAÇÃO



$$d: y = y_v - p/2$$

$$ax + by + c = 0$$

$\begin{matrix} 1 & -y_v + p/2 \end{matrix}$

$$d(P, F) = d(P, d)$$

$$\sqrt{(x - x_v)^2 + (y - y_v - p/2)^2} = \frac{|ax + by + c|}{\sqrt{a^2 + b^2}}$$

$$\sqrt{(x - x_v)^2 + (y - y_v - p/2)^2} = |y - y_v + p/2|$$

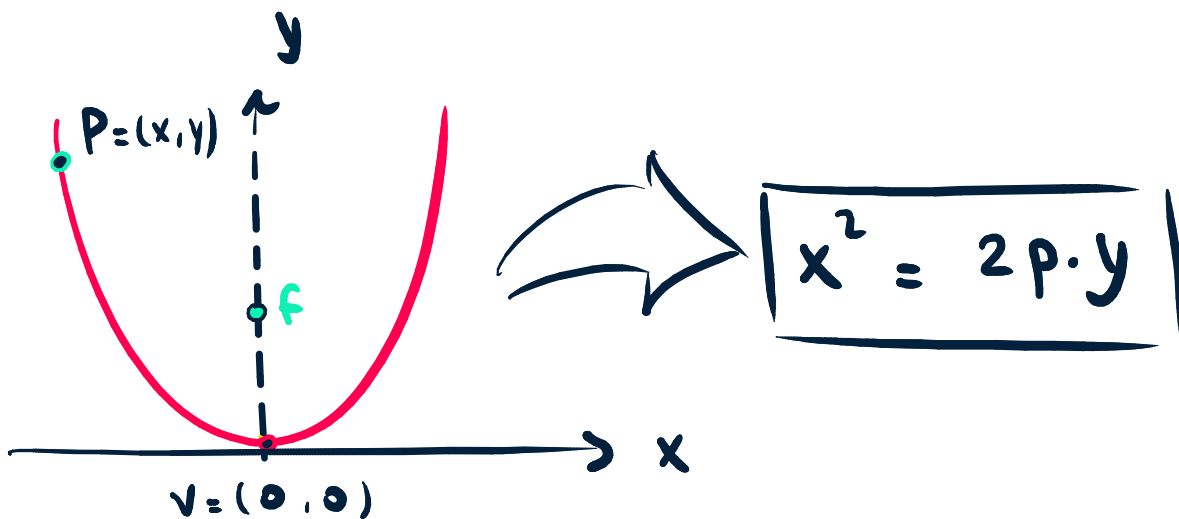
$$(x - x_v)^2 + (y - y_v - p/2)^2 = (y - y_v + p/2)^2$$



$$(x-x_v)^2 + (y-y_v - p/2)^2 - (y-y_v + p/2)^2 = 0$$

$$(x-x_v)^2 - 2p(y-y_v) = 0$$

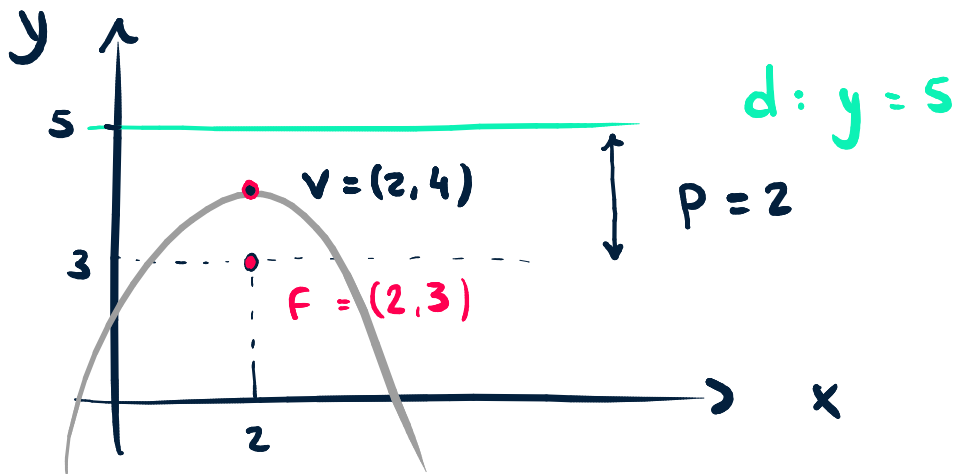
$$(x-x_v)^2 = 2p(y-y_v)$$



EXEMPLO 01

Seja $F = (2, 3)$ e $d: y = 5$,
encontre a equação da parábola.

R :



$$(x - x_v)^2 = -2p \cdot (y - y_v)$$

$$(x - 2)^2 = -4 \cdot (y - 4)$$



EXEMPLO 02 (ITA 2009)

Seja a parábola $2x^2 - 4x - 4y + 3 = 0$. Quanto vale a distância entre F e V?

$$\underline{R}: \quad P/2 = \underline{\underline{1/2}}$$

$$2(\underbrace{x^2 - 2x + 1}_{(x-1)^2}) + 3 = 4y + 2 \cdot 1$$

$$2(x-1)^2 = 4y - 1$$

$$(x-1)^2 = 2y - \frac{1}{2}$$

$$(x-1)^2 = 2\left(y - \frac{1}{4}\right)$$

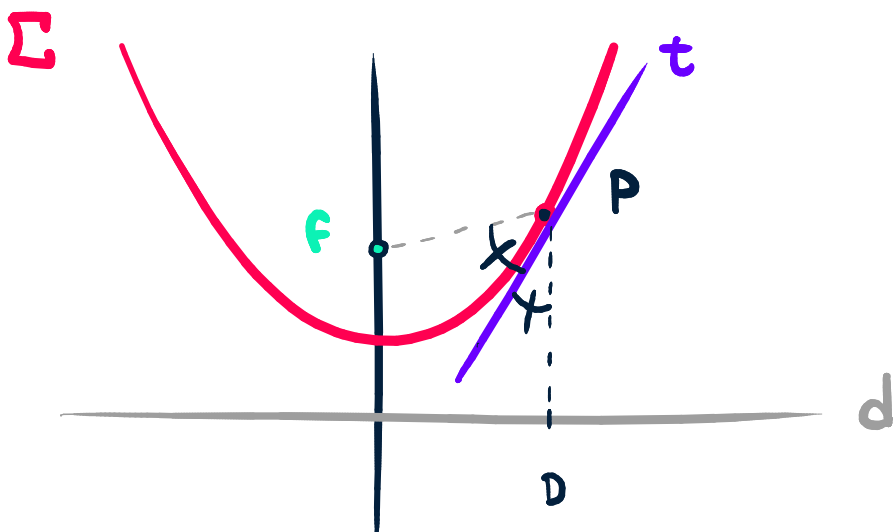
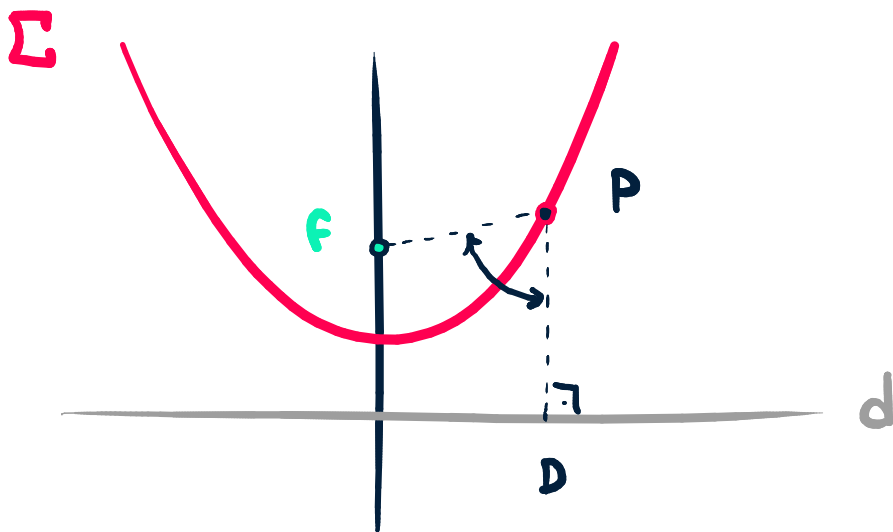
$$(x-x_v)^2 = 2p(y-y_v)$$

$$2p = 2 \therefore p = 1$$



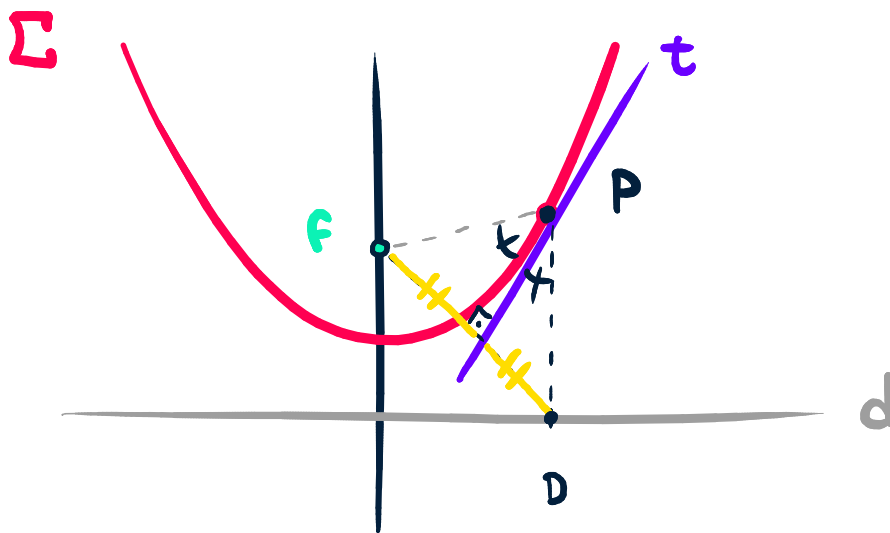
PARÁBOLA: TEOREMA DAS TANGENTES

Seja a parábola Σ de foco F e diretriz d , com $P \in \Sigma$ e D a projeção de P em d . A tangente t em P é bissetriz de $\widehat{FPD}$.

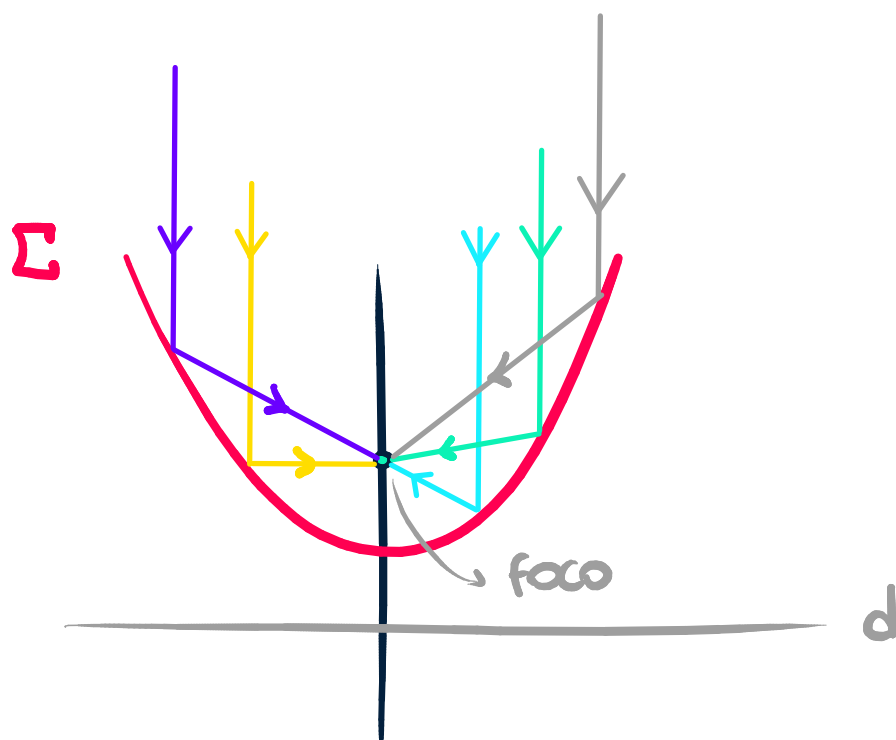
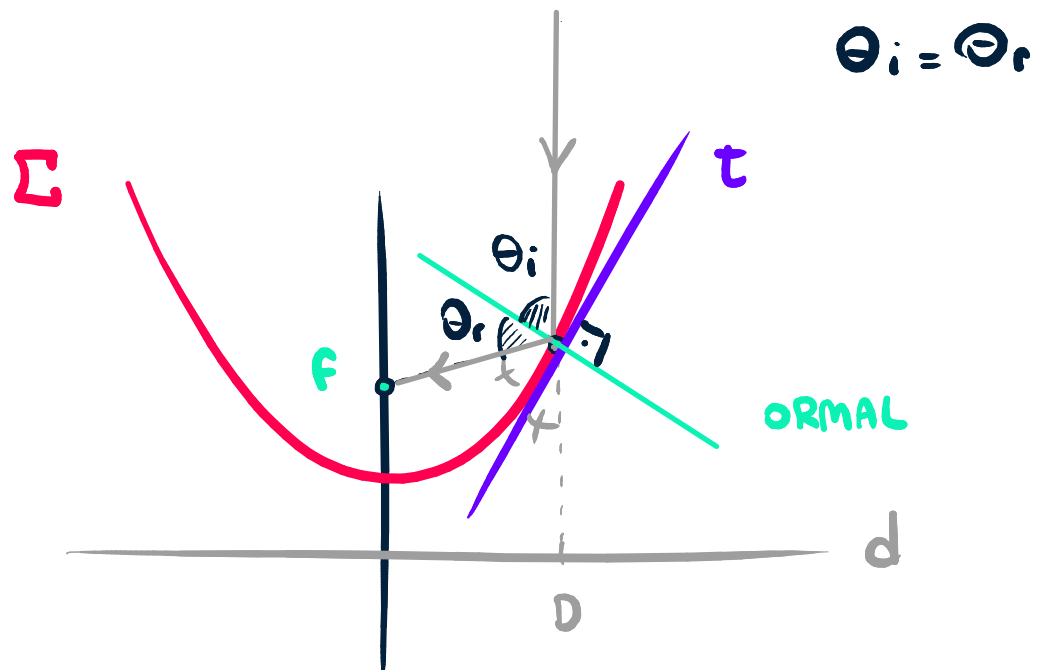


Consequências :

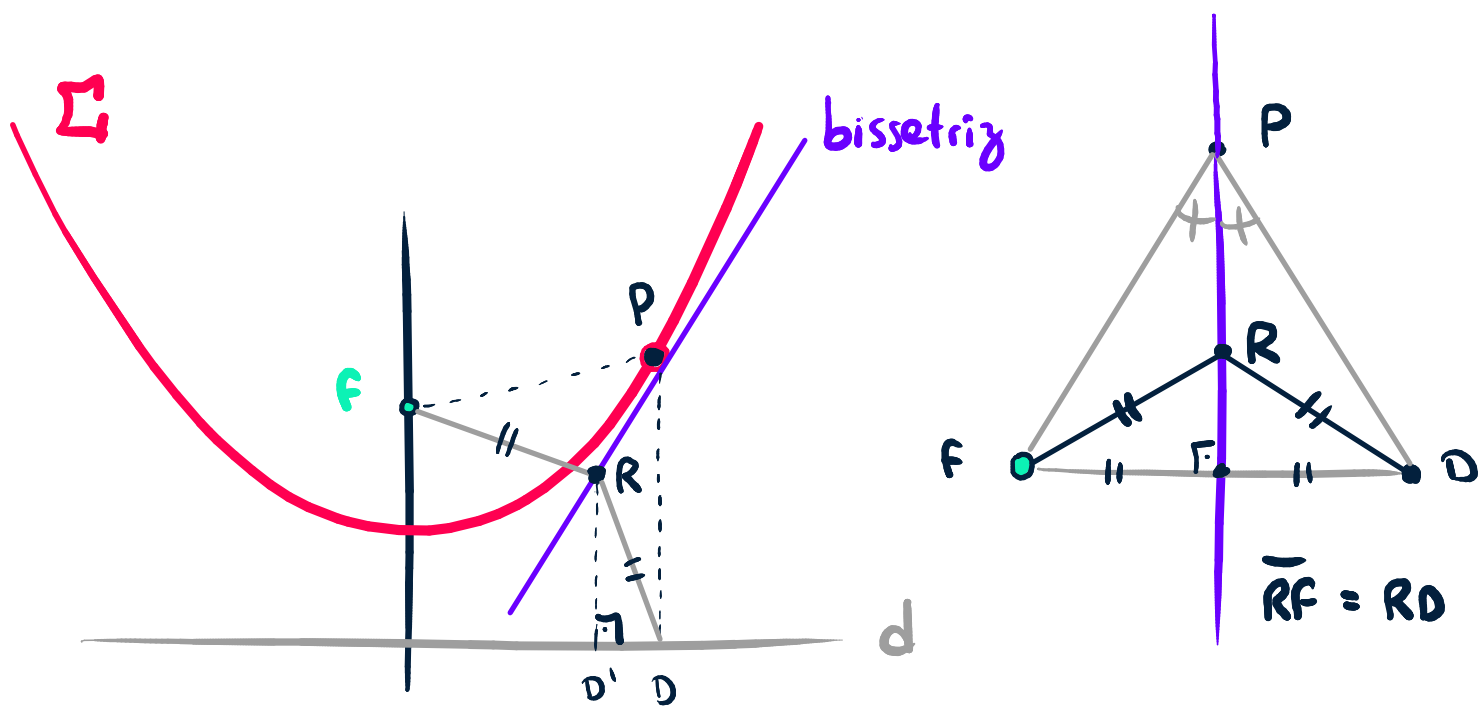
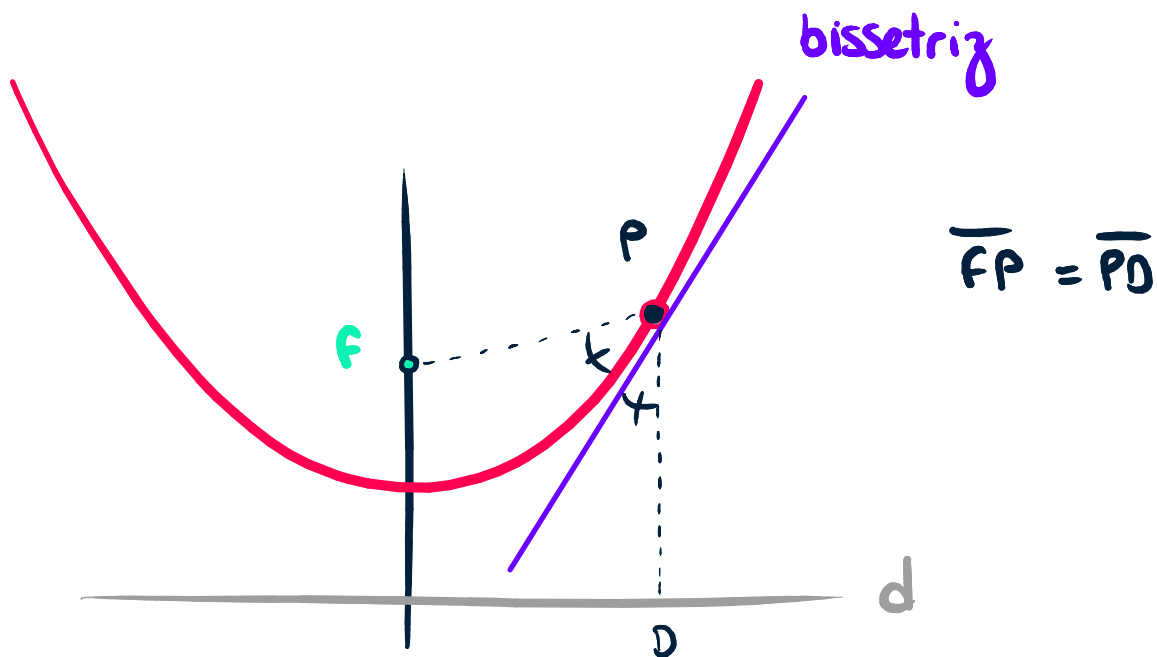
i) D é simétrico a F em relação a t .



ii) Todo raio paralelo ao eixo de simetria reflete passando pelo foco.



DEMONSTRAÇÃO



Se $R \in \Sigma$ então $\overline{RF} = \overline{RD'}$ (X)

$\overline{RF} > \overline{RD'}$ (✓)

