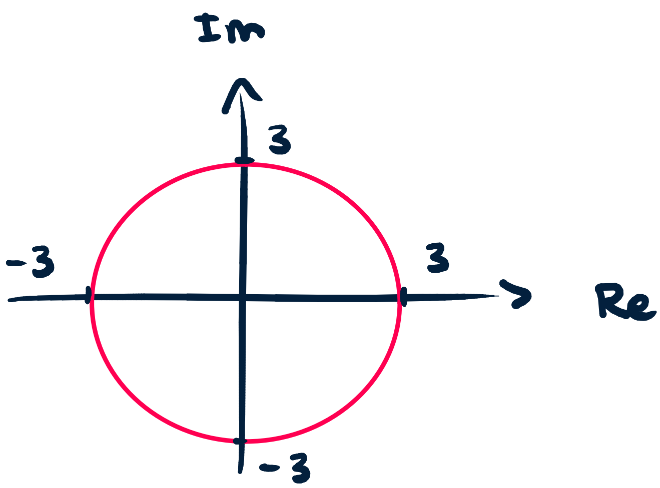
LUGARES GEOMÉTRICOS

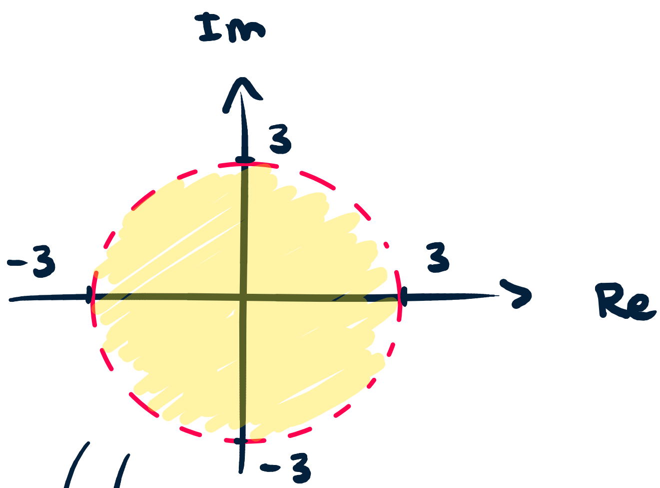
INTRODUÇÃO

Qual é a área da região definida por

$|z| < 3$, sendo $z \in \mathbb{C}$?

R :





$$A = \pi \cdot R^2$$

$$= \pi \cdot 3^2 = 9\pi$$

EXEMPLO 01

Esboce, no plano complexo, a região definida por:

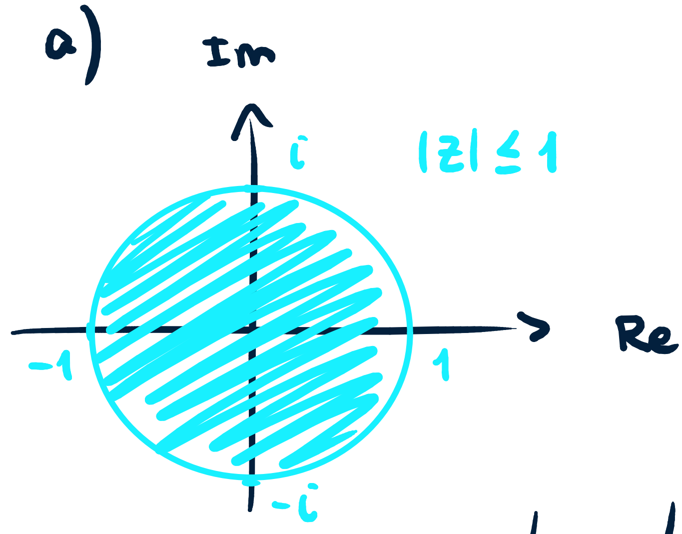
a) $Re(z) \leq 0$ e $|z| \leq 1$

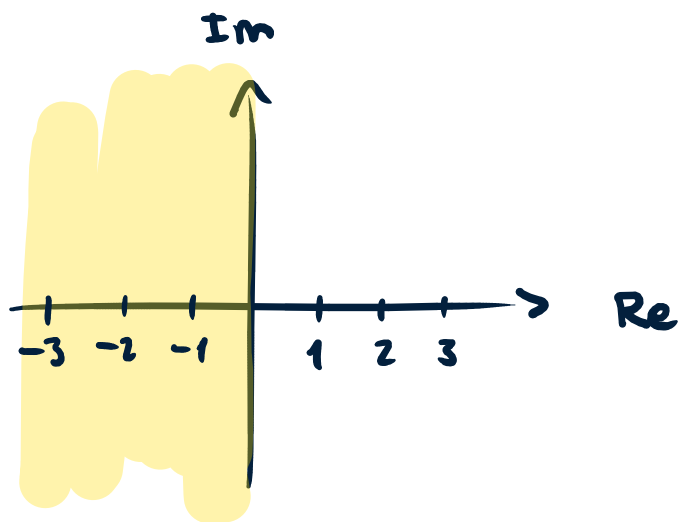
b) $2 \leq |z| \leq 4$ e $Im(z) < 1$

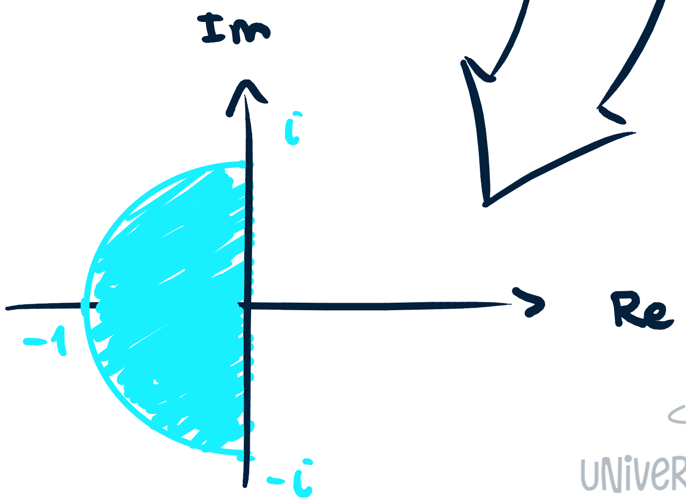
c) $|z-3| \leq 1$ e $Re(z) \geq 3$

R :

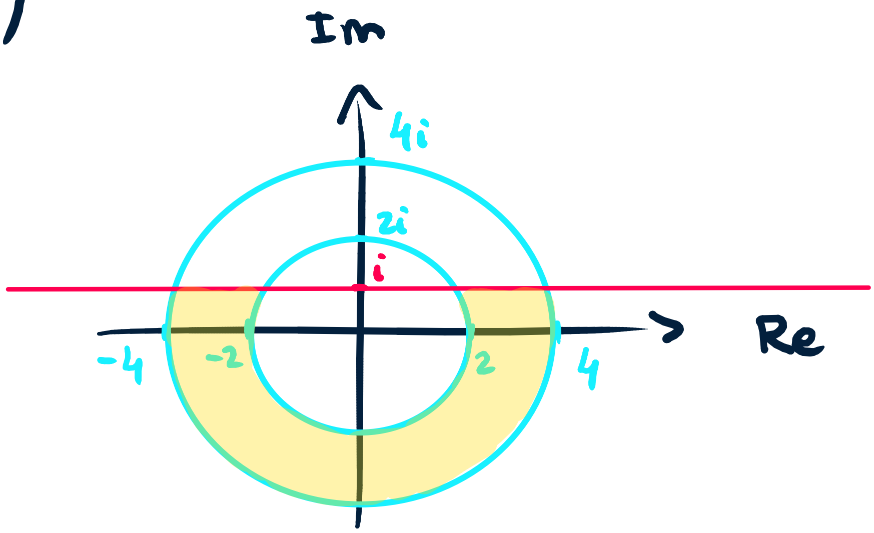
a)



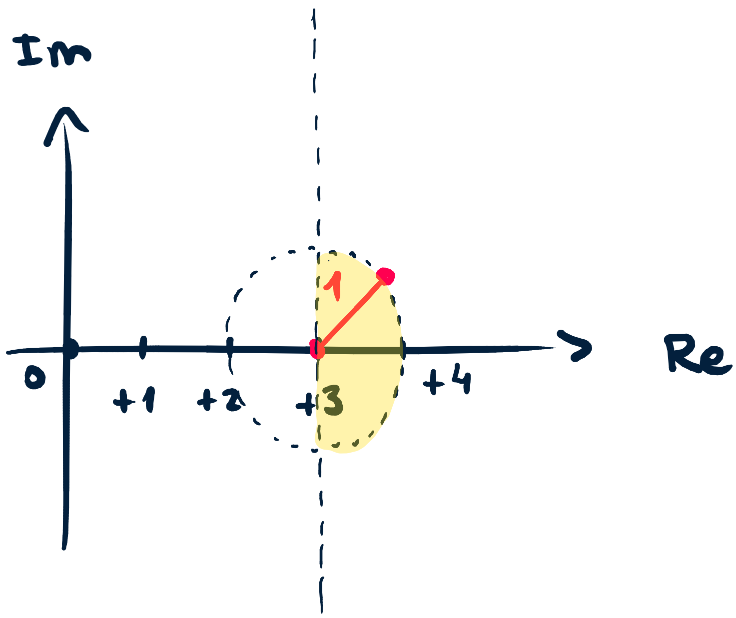




b)



c)

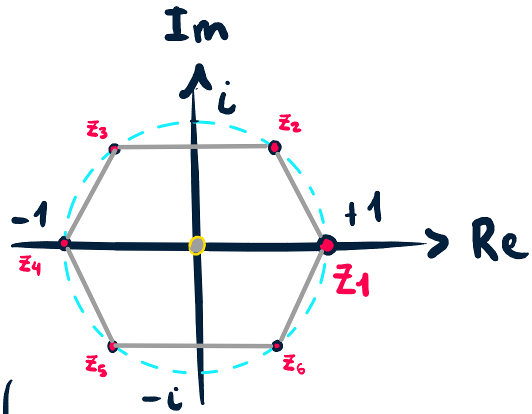


EXEMPLO 02 (ITA 1997)

Seja um hexágono regular de centro $z_0 = i$, e os vértices dados por z_n, $n \in \mathbb{N} \mid 1 \leq n \leq 6$.

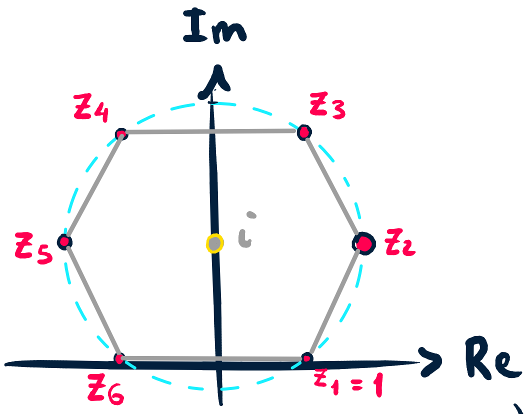
Se $z_1 = 1$, calcule $2z_3$.

R:



z_n : soluções de $(z - 0)^6 = 1$

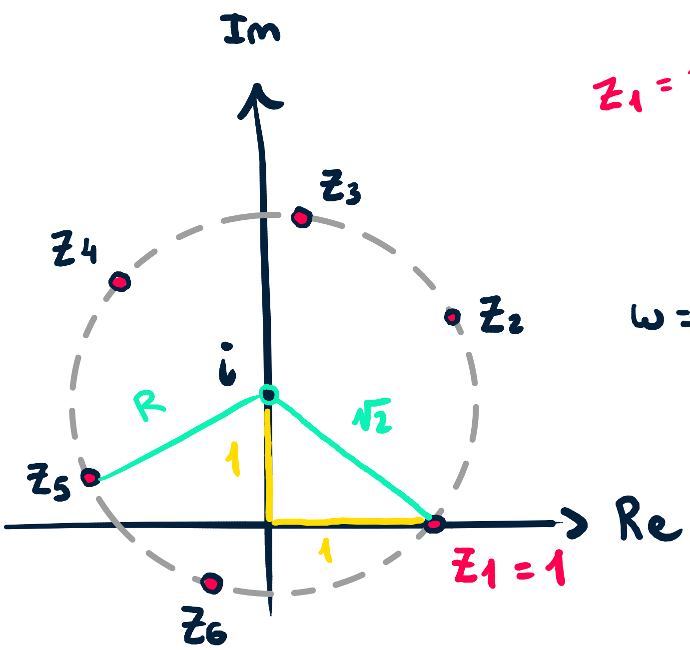
CENTRO DO HEXÁGONO



z_n : soluções de $(z - i)^6 = \omega$

CENTRO DO HEXÁGONO

UNIVERSO NARRADO



$$z_1 = 1 \quad (z - i)^6 = w \;\longrightarrow\; (1 - i)^6 = w$$

$$w = \left[(1-i)^2\right]^3 = \left[\cancel{1} - 2i + \cancel{i^2}\right]^3$$

$$w = -8 \cdot \underbrace{i^3}_{-i} = 8i$$

$$(z - i)^6 = 8i = 8\left[\cos \pi/2 + i \operatorname{sen} \pi/2\right]$$

$$z - i = 8^{1/6}\left[\cos\left(\frac{\pi/2 + 2k\pi}{6}\right) + i \operatorname{sen}\left(\frac{\pi/2 + 2k\pi}{6}\right)\right]$$

$$z_n = i + \sqrt{2}\cos\underbrace{\left(\frac{\pi/2 + 2k\pi}{6}\right)}_{\sqrt{2}/2} + i\sqrt{2}\operatorname{sen}\underbrace{\left(\frac{\pi/2 + 2k\pi}{6}\right)}_{-\sqrt{2}/2}$$

$$z_1 = 1 \Rightarrow \frac{\pi/2 + 2k\pi}{6} = 2\pi - \pi/4 \;\therefore\; \boxed{k = 5}$$

$$z_3 \rightarrow k = 7 \text{ ou } k = 1$$

$$2z_3 = 2i + 2\sqrt{2}\cos\left(\frac{5\pi}{12}\right) + 2i\sqrt{2}\operatorname{sen}\left(\frac{5\pi}{12}\right)$$

$$2z_3 = 2i + 2\sqrt{2}\cos\left(\frac{5\pi}{12}\right) + 2i\sqrt{2}\,\mathrm{sen}\left(\frac{5\pi}{12}\right)$$

$$2z_3 = 2i + 2\sqrt{2}\,\frac{(\sqrt{6}-\sqrt{2})}{4} + 2i\sqrt{2}\,\frac{(\sqrt{2}+\sqrt{6})}{4}$$

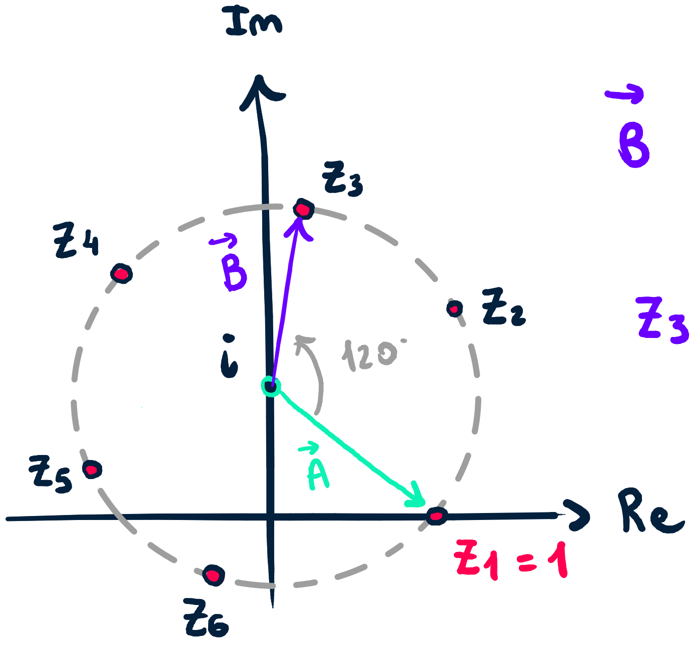
$$= 2i + \frac{i\sqrt{2}(\sqrt{2}+\sqrt{6})}{2} + \frac{\sqrt{2}(\sqrt{6}-\sqrt{2})}{2}$$

$$= 2i + \frac{i(\cancel{2}+\cancel{2}\sqrt{3})}{\cancel{2}} + \frac{\cancel{2}\sqrt{3}-\cancel{2}}{\cancel{2}}$$

$$= 2i + i(1+\sqrt{3}) + \sqrt{3}-1$$

$$= \underline{(3+\sqrt{3})i + (\sqrt{3}-1)}\ //$$

UNIVERSO NARRADO



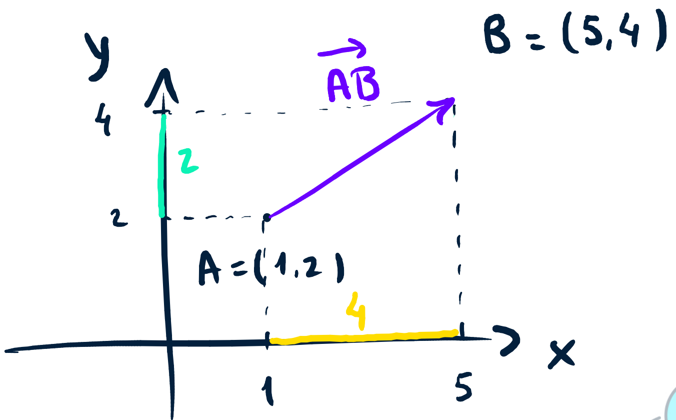
$$\vec{B} = \vec{A}\left[\cos\left(2\pi/3\right) + i \operatorname{sen}\left(2\pi/3\right)\right]$$

$$z_3 - i = (1 - i)\left[-\frac{1}{2} + \frac{\sqrt{3}\, i}{2}\right]$$

$$2z_3 = 2i + (1-i)\frac{(i\sqrt{3} - 1)}{\cancel{2}} \cdot \cancel{2}$$

$$2z_3 = 2i + i\sqrt{3} - 1 + \sqrt{3} + i$$

$$= \underline{(3 + \sqrt{3})i + (\sqrt{3} - 1)} \,//$$



$$\vec{AB} = B - A$$

$$= (x, y)$$

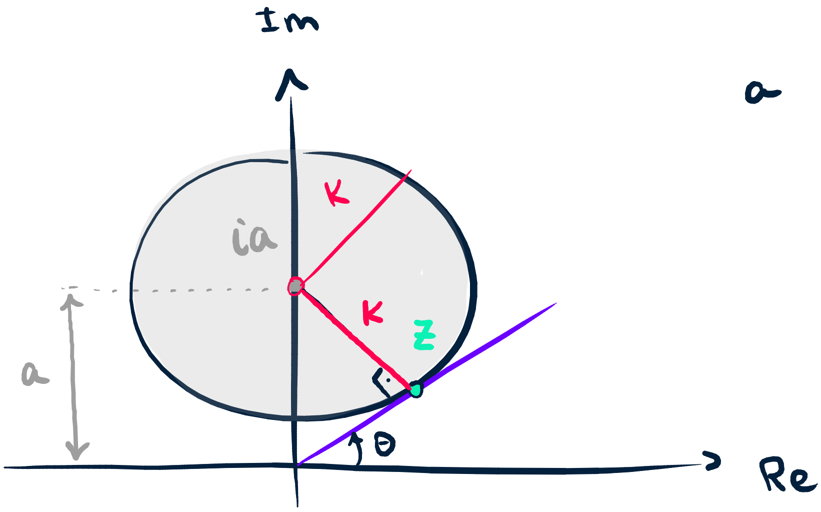
$$= (4, 2)$$

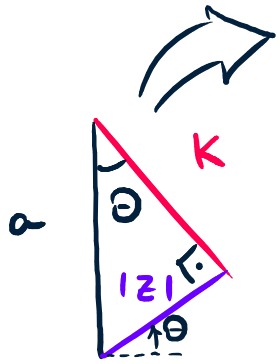
EXEMPLO 03 (ITA)

Determine $z \in \mathbb{C}$ com o menor argumento tal que

$$|z - ia| \leq K$$

R:





$$\cos\theta = \frac{|z|}{a}$$

$$\operatorname{sen}\theta = \frac{K}{a}$$

$$a^2 = K^2 + |z|^2$$

$$|z| = \sqrt{a^2 - K^2}$$

$$z = |z|\left[\cos\theta + i\operatorname{sen}\theta\right]$$

$$= |z|\left[\frac{|z|}{a} + i\,\frac{K}{a}\right]$$

$$\boxed{z = \frac{(a^2 - K^2)}{a} + \frac{K\sqrt{a^2 - z^2}}{a}\,i}$$

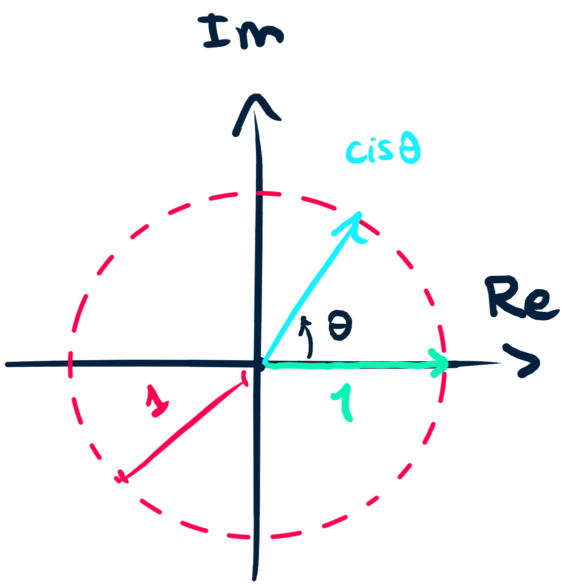
EXEMPLO 04

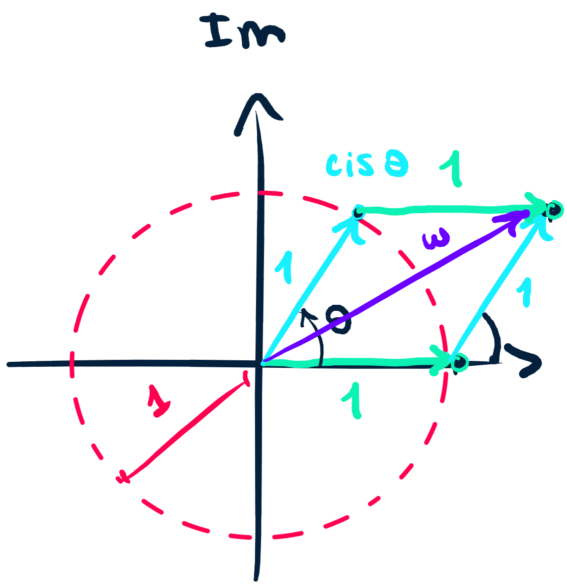
Mostre que $1+\text{cis}\,\theta = 2\cos(\theta/2)\,\text{cis}(\theta/2)$

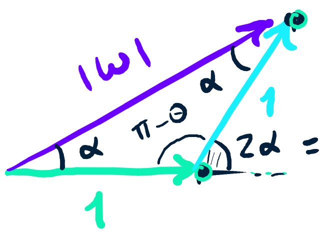
$$1+\cos\theta + i\,\text{sen}\,\theta = 2\cos(\theta/2)\left[\cos(\theta/2) + i\,\text{sen}(\theta/2)\right]$$

R :

$$1+\cos\theta + i\,\text{sen}\,\theta = \omega$$







$2\alpha = \theta \quad \therefore \quad \alpha = \theta/2$

$$|\omega|^2 = 1^2 + 1^2 - 2.1.1.\cos(\pi-\theta)$$

$$= 2 - 2(-\cos\theta)$$

$$|\omega|^2 = 2(1+\cos\theta) = 4\cos^2(\theta/2)$$

$$|\omega| = 2\cos(\theta/2)$$

$$\omega = |\omega|(\cos\alpha + i\,\text{sen}\,\alpha)$$

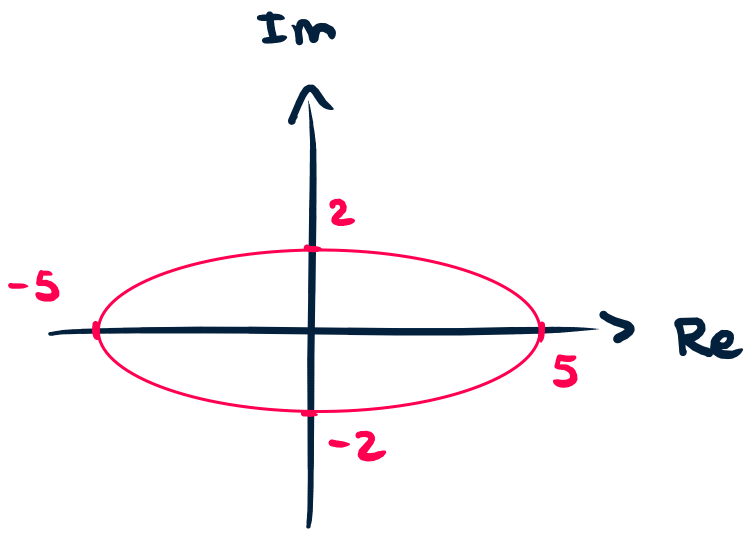
↳ $\omega = 2\cos(\theta/2).\text{cis}(\theta/2)$

EXEMPLO 05

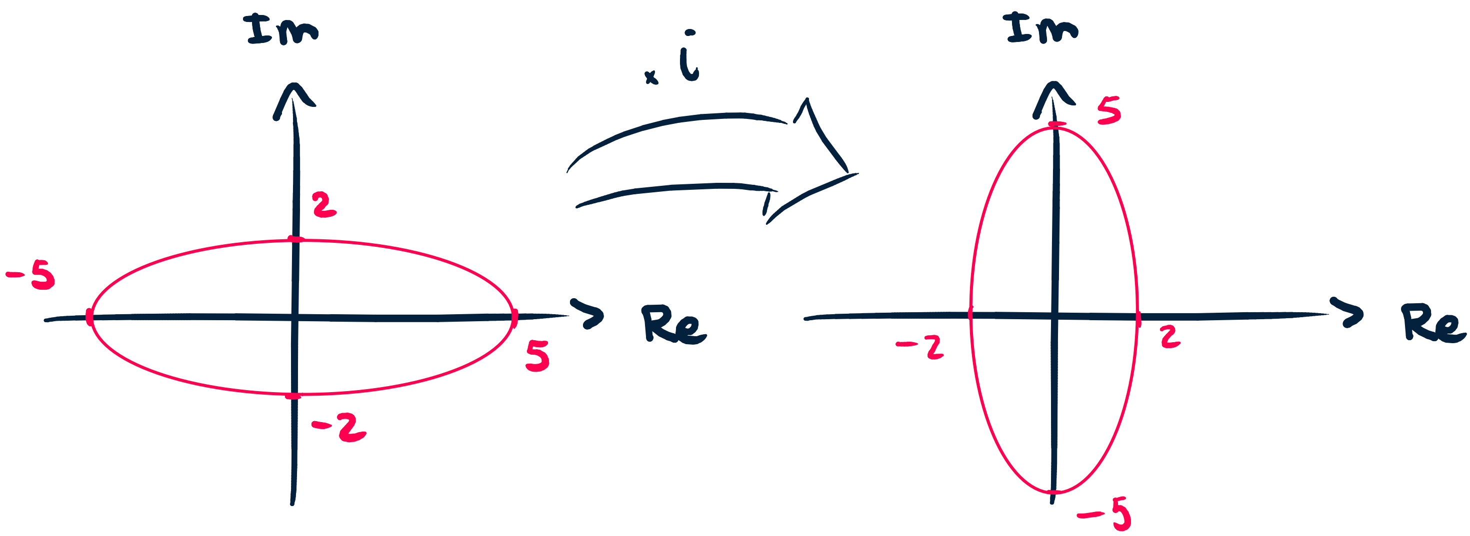
Seja a função $g: \mathbb{C} \to \mathbb{C}$ dada por

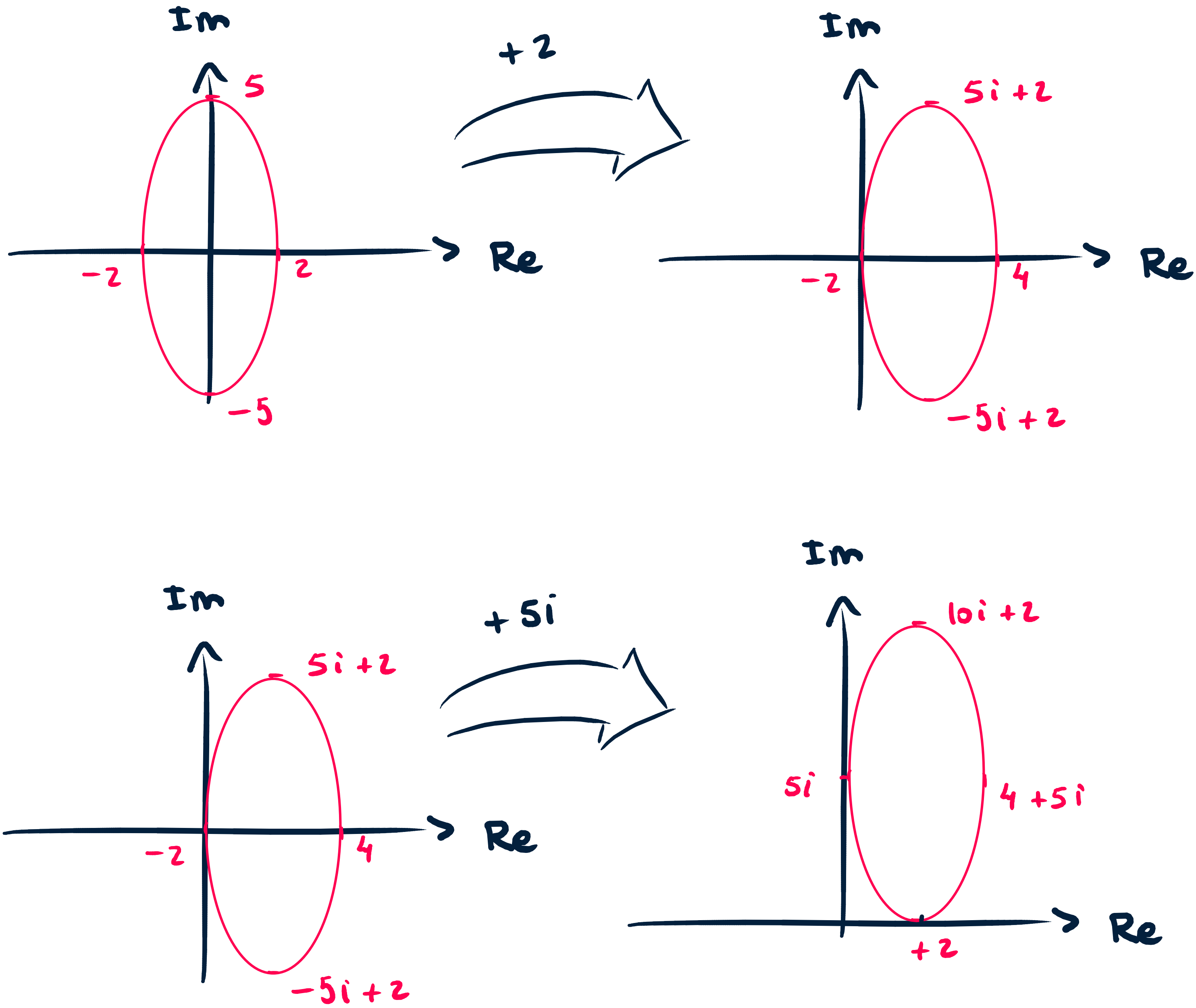
$g(z) = z \cdot i + 2 + 5i$.

Seja A a elipse abaixo. Esboce $g(A)$.



R:



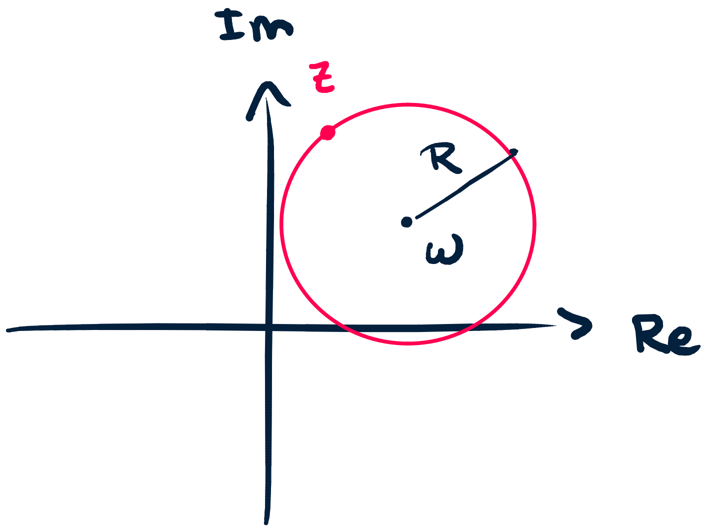


Im
5
-2
2
Re
-5
+2
Im
5i +2
-2
4
Re
-5i +2
Im
5i +2
-2
4
Re
-5i +2
+5i
Im
10i +2
5i
4 +5i
Re
+2

CÔNICAS

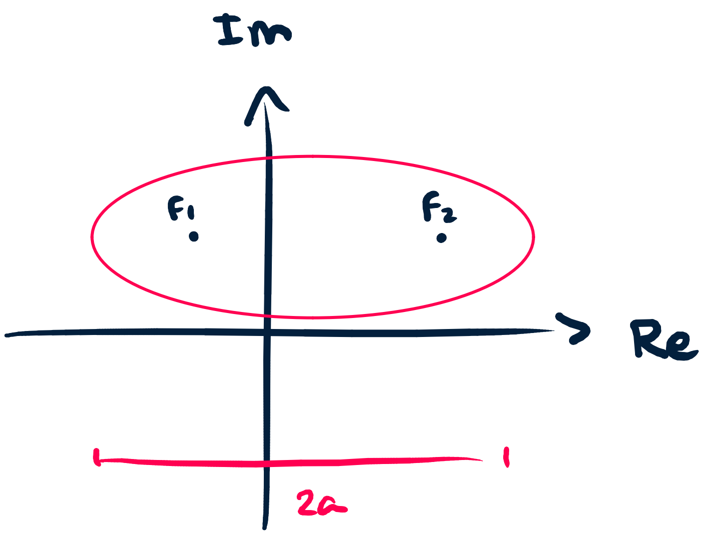
CIRCUNFERÊNCIA

$$|z - w| = R$$



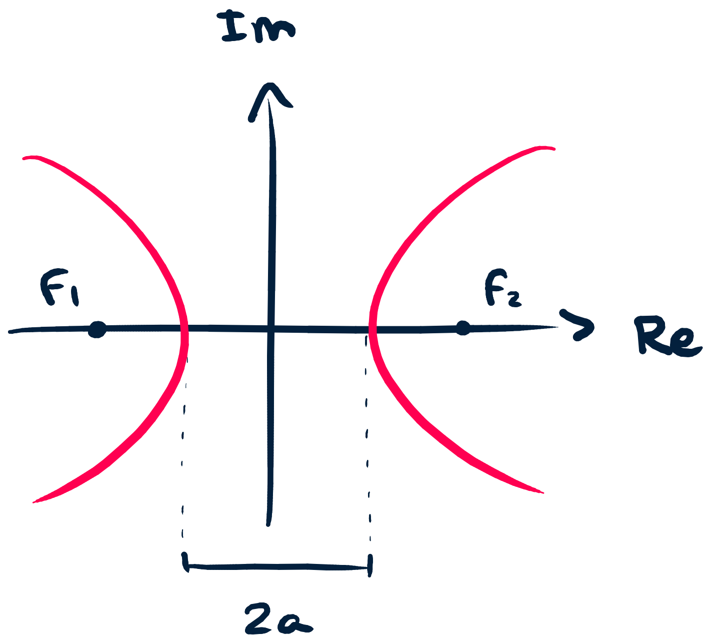
ELIPSE

$$|z - F_1| + |z - F_2| = 2a$$



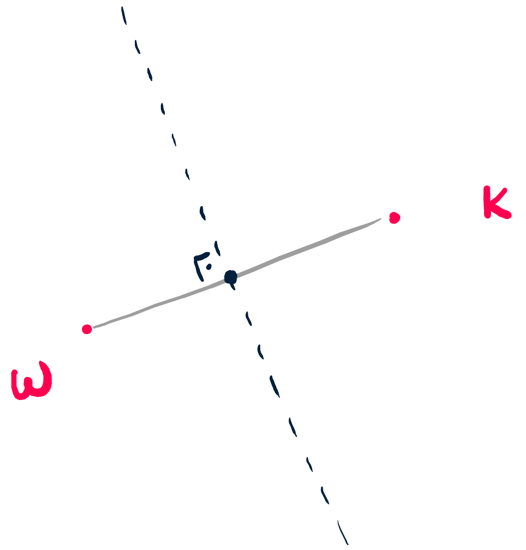
HIPÉRBOLE

$$|z - F_1| - |z - F_2| = 2a$$

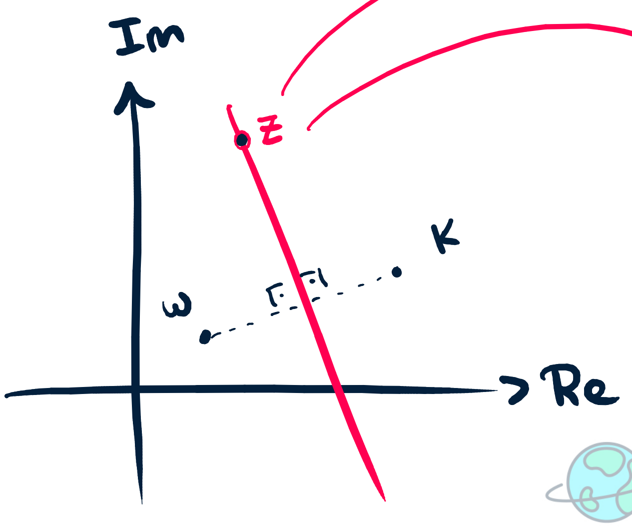


MEDIATRIZ

↳ Lugar geométrico dos pontos equidistantes de dois pontos dados.



No plano complexo:

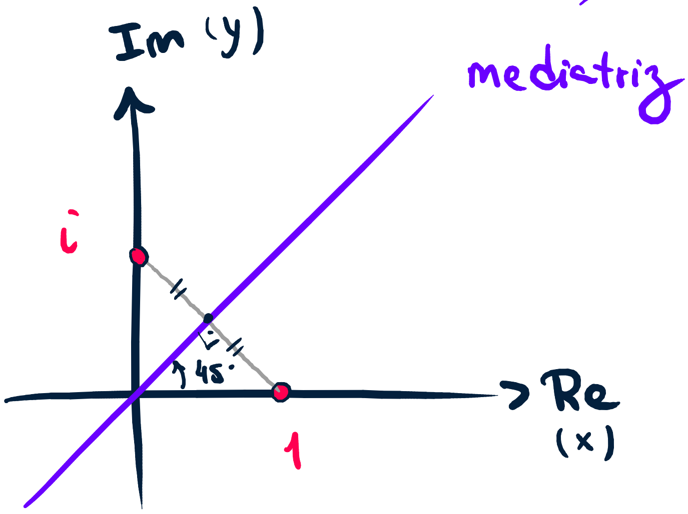


$$\boxed{|z - w| = |z - k|}$$

Determine a equação da mediatriz dos pontos

$w = i$ e $K = 1$

R:



$|z - i| = |z - 1|$

$z = x + yi$

$y = x$

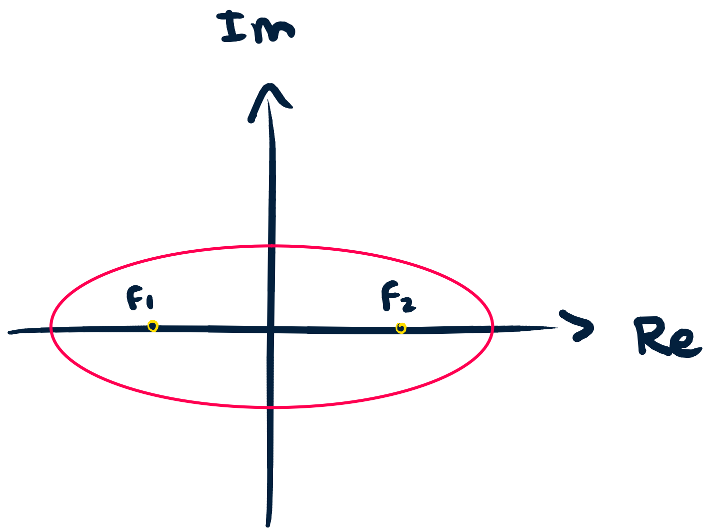
$z = x + xi$

EXEMPLO 01

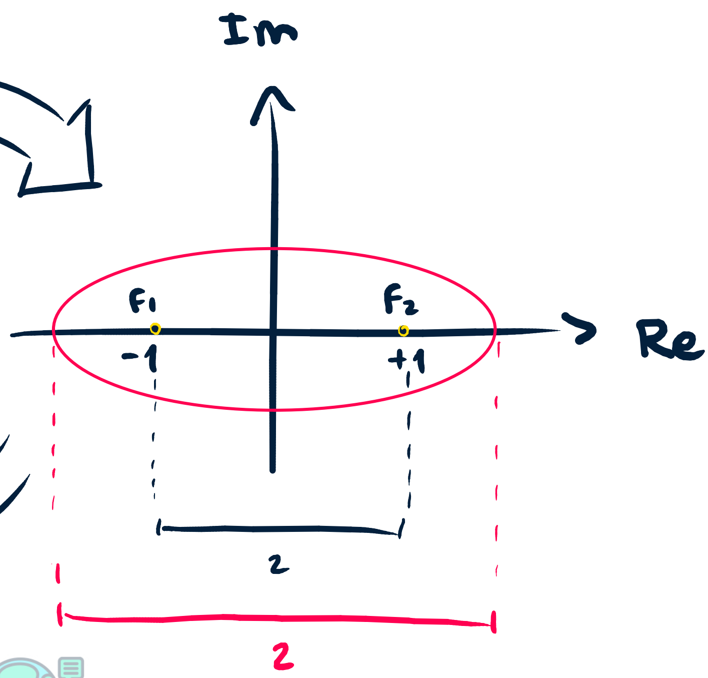
Localize no plano complexo o lugar geométrico dos nºs $z \in \mathbb{C}$ tais que $|z-1|+|z+1| = 2$.

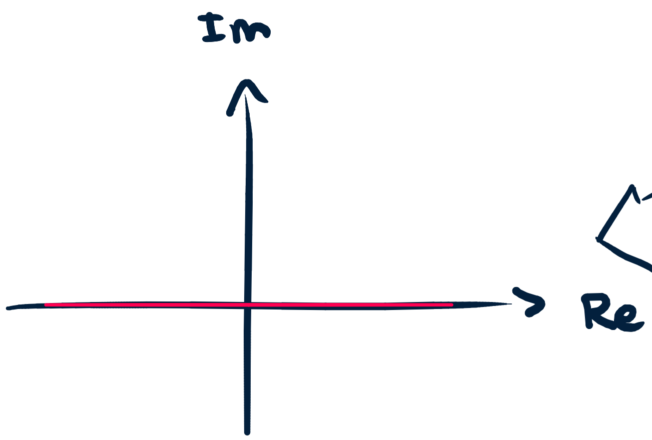
R:

$$|z-F_1| + |z-F_2| = 2a$$



$$|z-1| + |z+1| = 2$$





EXEMPLO 02 (IME 1970)

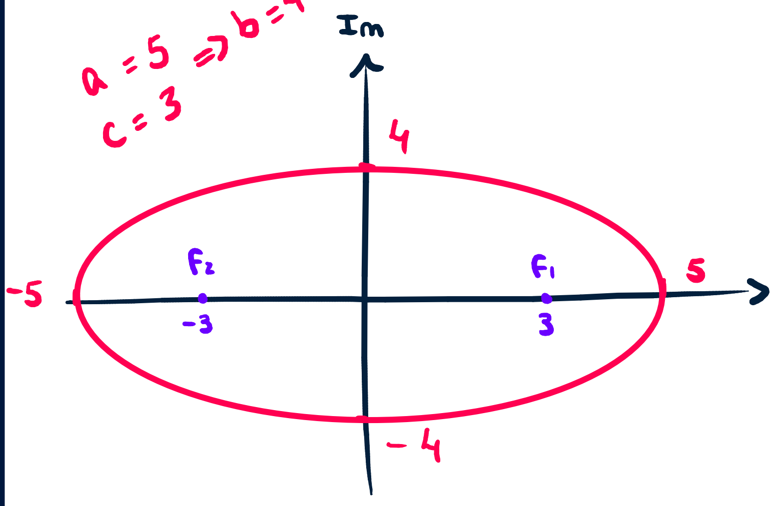
Determine a solução do sistema

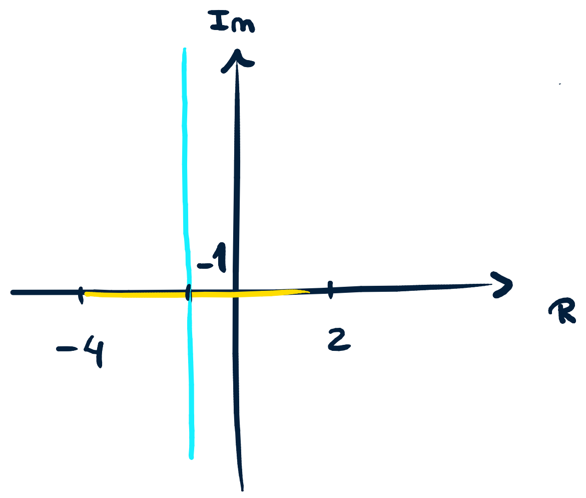
$$\begin{cases} |z-2| = |z+4| \\ |z-3| + |z+3| = 10 \end{cases}$$

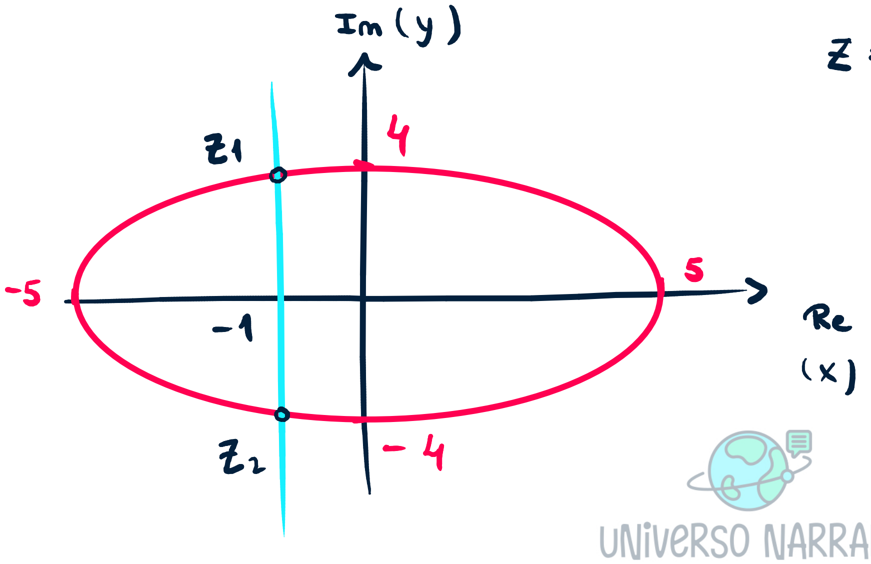
R :

$$\begin{cases} |z-3| + |z+3| = 2 \cdot 5 \\ |z-2| = |z+4| \end{cases} \dashrightarrow \begin{array}{l} a^2 = b^2 + c^2 \\ 5^2 = 4^2 + 3^2 \end{array}$$

$a = 5$ $\Rightarrow b = 4$
$c = 3$







$z = x + yi$

$$\begin{cases} x = -1 \\ \dfrac{x^2}{5^2} + \dfrac{y^2}{4^2} = 1 \end{cases}$$

$$x = -1 \Longrightarrow \frac{(-1)^2}{25} + \frac{y^2}{16} = 1$$

$$y^2 = \left(1 - \frac{1}{25}\right) \cdot 16 = \left(\frac{25-1}{25}\right) 16 = \frac{24}{25} \cdot 16$$

$$y = \pm \frac{4}{5} \cdot 2\sqrt{6} = \pm \frac{8\sqrt{6}}{5}$$

$$z_1 = -1 + \frac{8\sqrt{6}}{5} i$$

$$z = x + yi$$

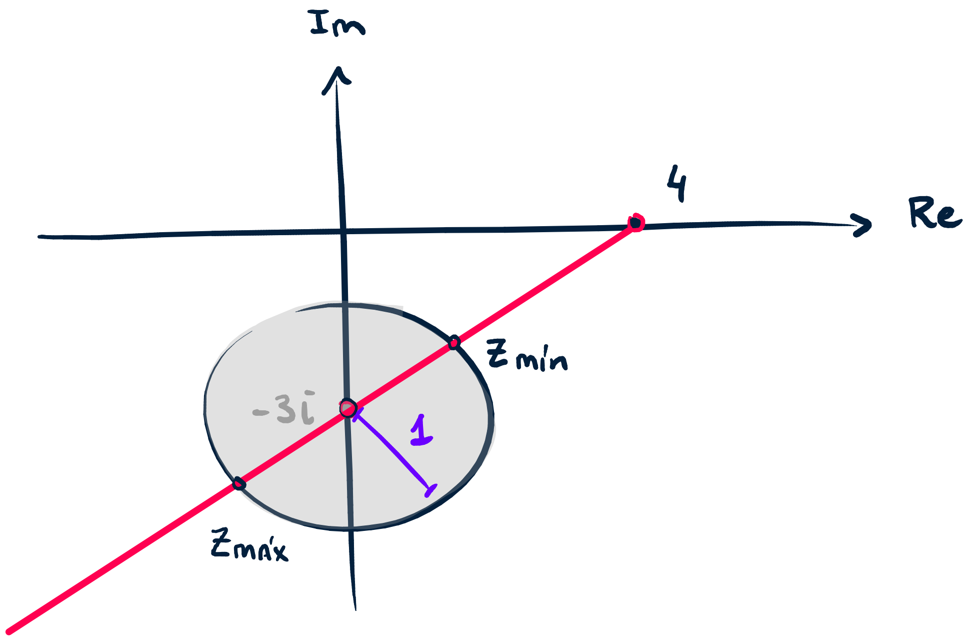
$$z_2 = -1 - \frac{8\sqrt{6}}{5} i$$

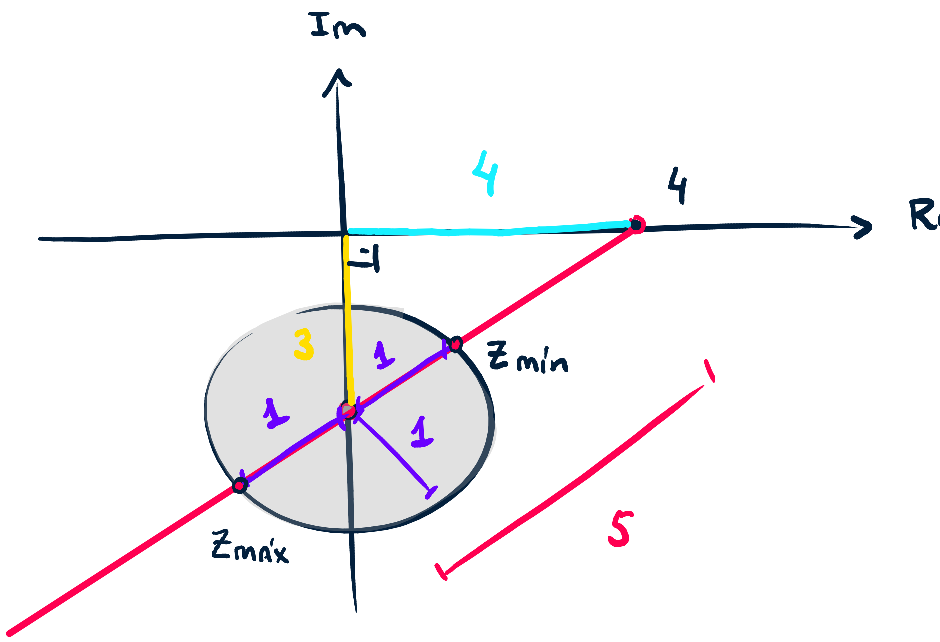
EXEMPLO 03 (IME)

Determine o maior e menor valor de $|z-4|$ tal que

$|z+3i| \leq 1$

R :





$|z-4|_{min} = 4$

$|z-4|_{máx} = 6$

EXEMPLO 04

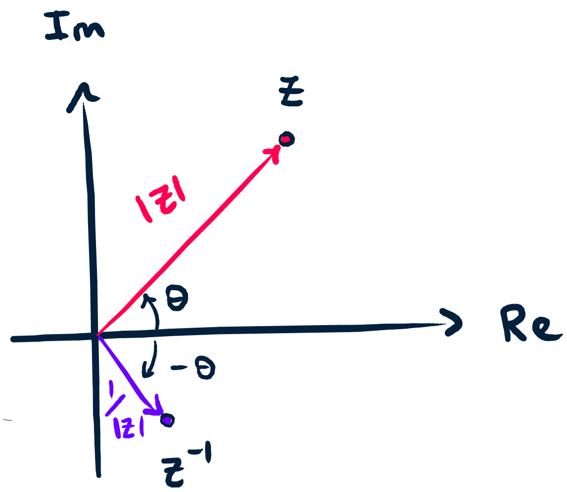
Determine o valor máximo de $|z|$ sabendo que

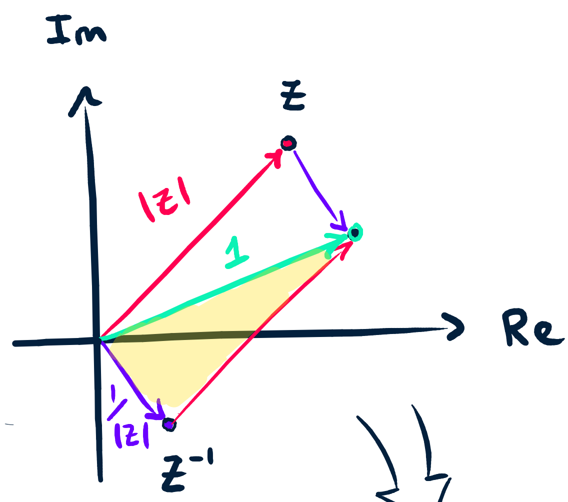
$|z + z^{-1}| = 1$.

R:

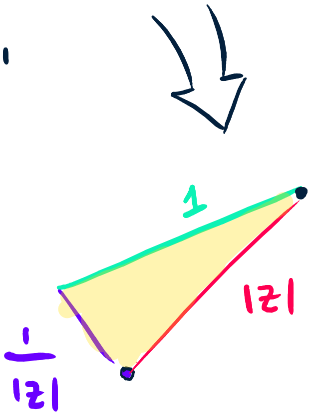
$$z = |z|\,[\cos\theta + i\,\mathrm{sen}\,\theta]$$

$$z^{-1} = \frac{1}{|z|}\,[\cos(-\theta) + i\,\mathrm{sen}(-\theta)]$$





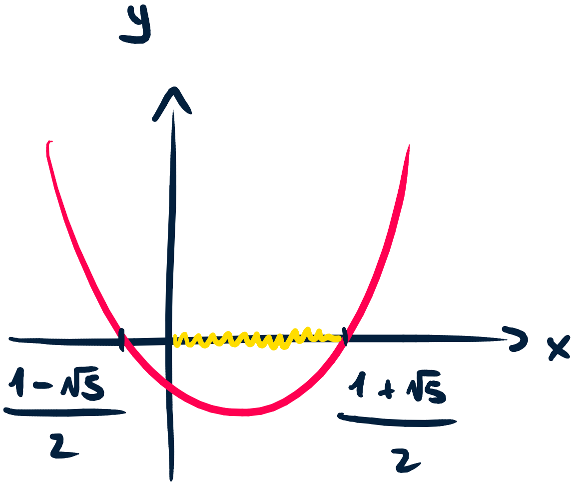
$$1 + \frac{1}{|z|} \geq |z| \;\therefore\; \boxed{|z|^2 - |z| - 1 \leq 0}$$



$$\boxed{|z|^2 - |z| - 1 \leq 0}$$

$$x^2 - x - 1 \leq 0 \qquad x \geq 0$$

$$x^2 - x - 1 = 0$$



$$\Delta = 1 - 4.(-1) = 5$$

$$x = \frac{1 \pm \sqrt{5}}{2}$$

$$0 \leq |z| \leq \frac{1+\sqrt{5}}{2}$$

$$\boxed{|z|_{máx} = \frac{1+\sqrt{5}}{2}}$$