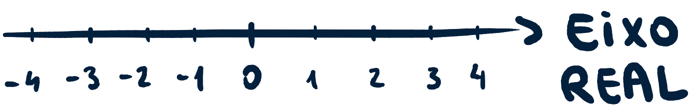
· FORMA ALGÉBRICA ·

O PLANO COMPLEXO

(ARGAND - GAUSS)



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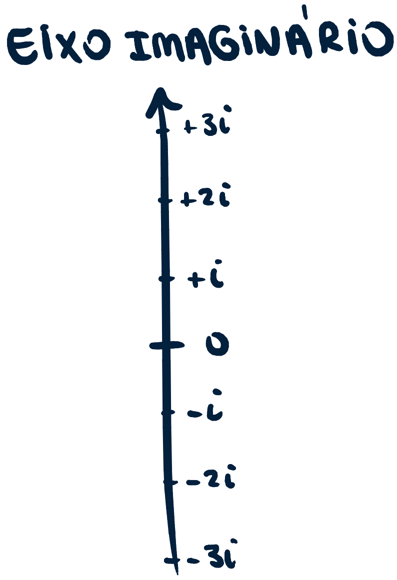
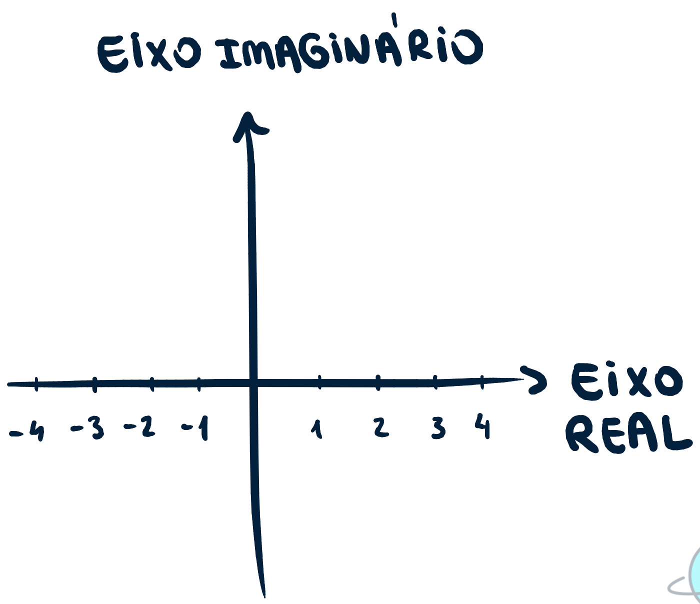
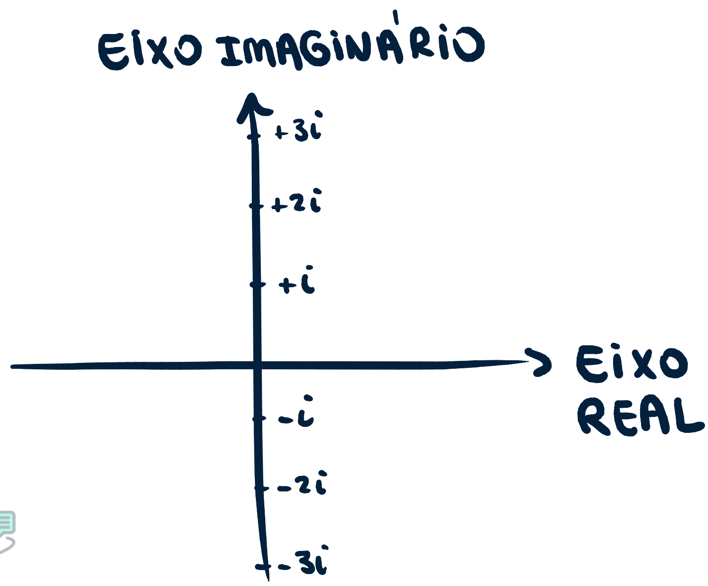
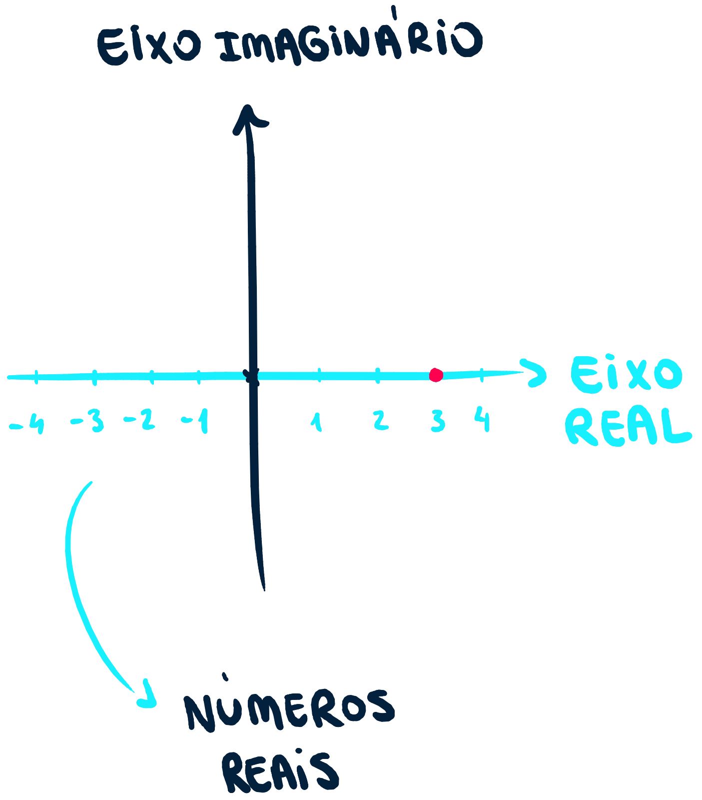
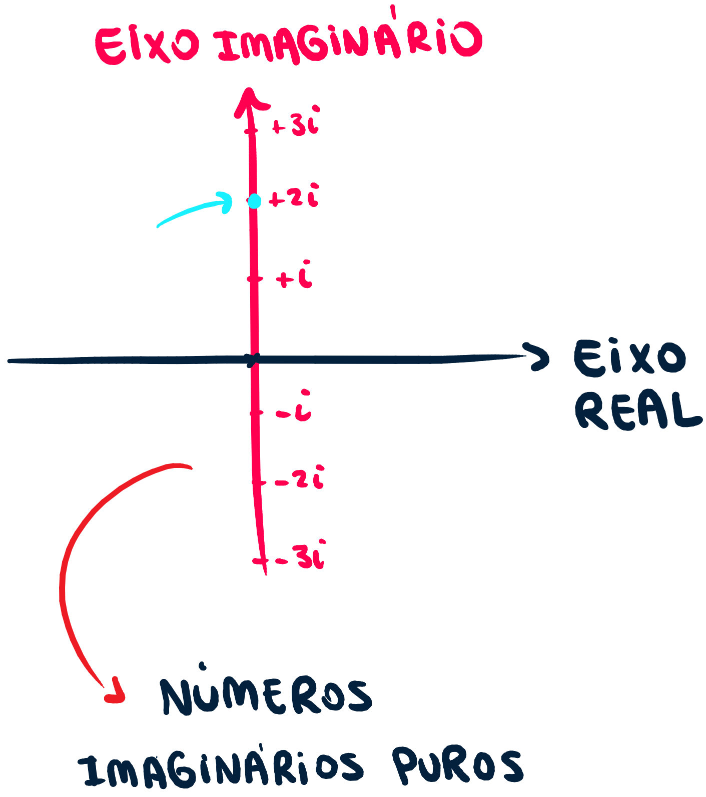


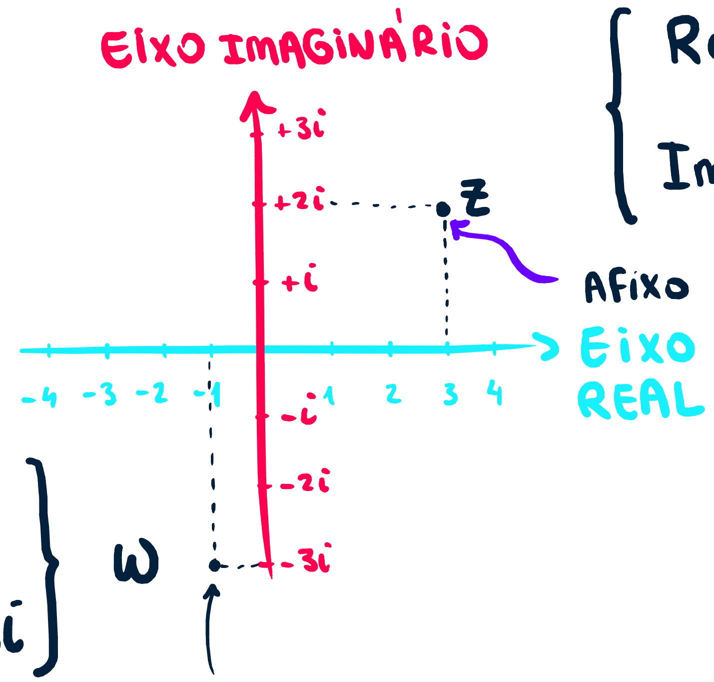








EM GERAL UM NÚMERO COMPLEXO PODE TER AS DUAS PARTES - REAL E IMAGINÁRIA:

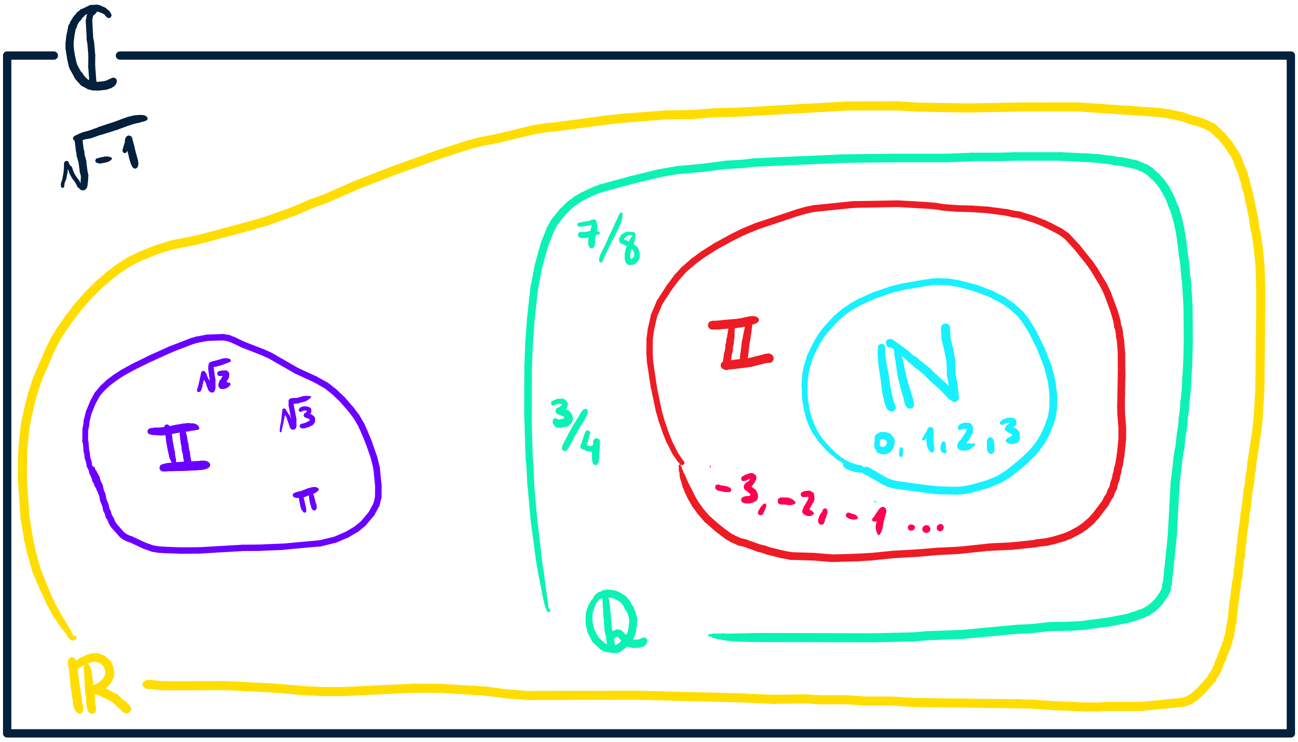


$$\begin{cases} Re(z) = 3 \\ Im(z) = 2i \end{cases}$$

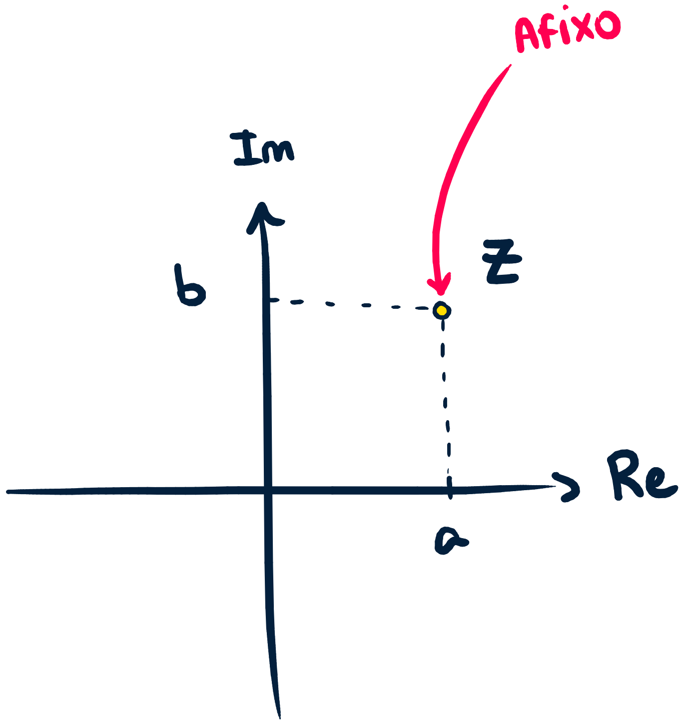
$\hookrightarrow z = (3, 2)$

$= (Re, Im)$

$$\left. \begin{array}{l} Re(w) = -1 \\ Im(w) = -3i \end{array} \right\} w$$

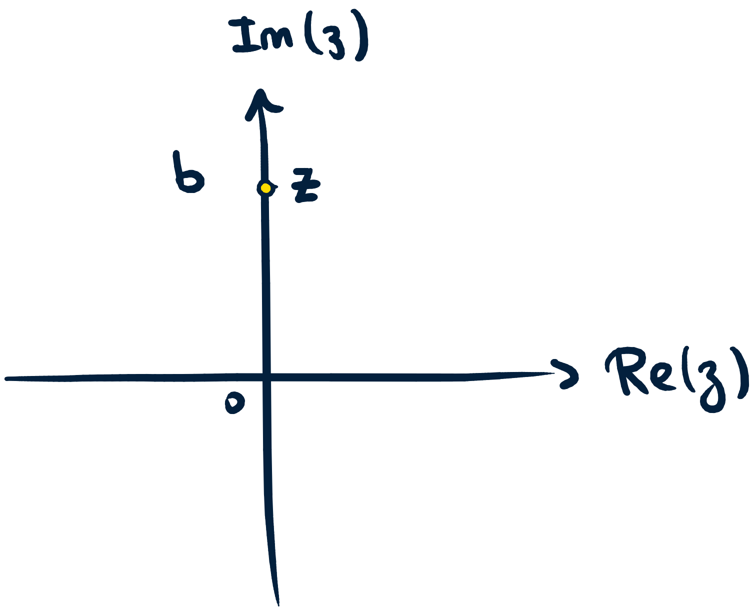

ℂ
√-1
ℝ
𝕀
√2
√3
π
ℚ
7/8
3/4
ℤ
-3, -2, -1 ...
ℕ
0, 1, 2, 3

· FORMA ALGÉBRICA DO NÚMERO COMPLEXO ·



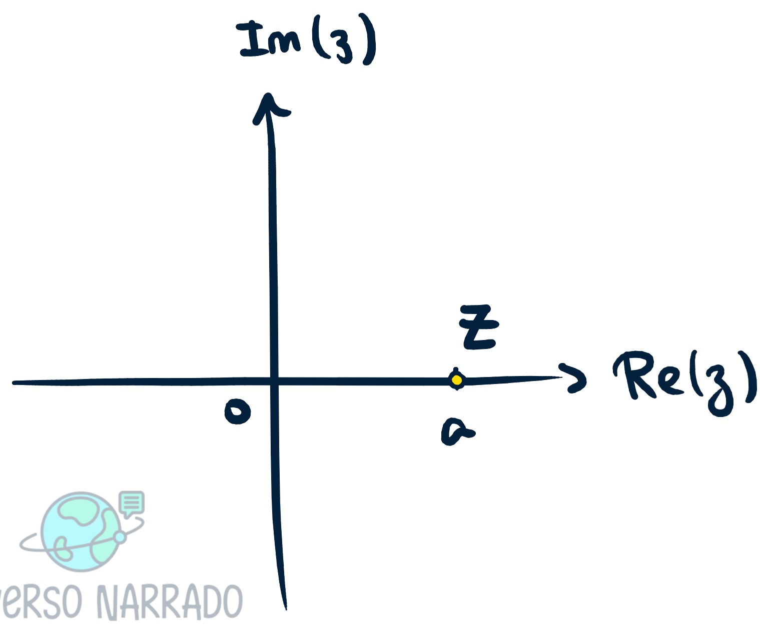
$$\boxed{z = a + bi}$$

$$z = (a, b)$$
$$= (Re, Im)$$



Se $a = 0$:

tem-se um número imaginário puro

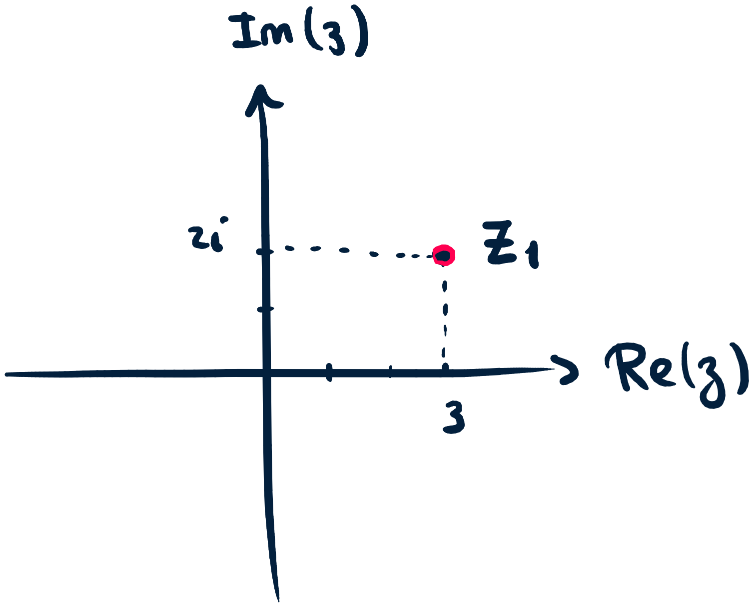


Se $b = 0$:

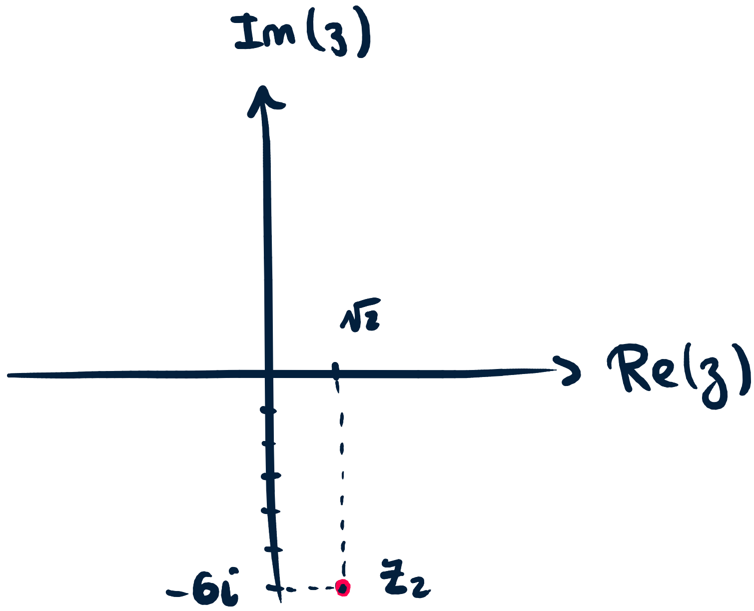
tem-se um número real

EXEMPLO

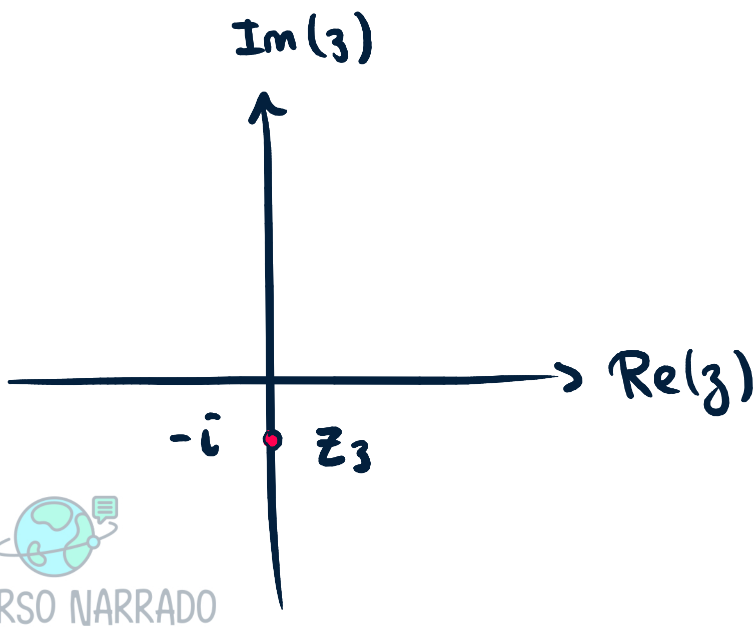
$z_1 = -3 + 2i$



$z_2 = \sqrt{2} - 6i$



$z_3 = -i$



SOMA DE NÚMEROS COMPLEXOS

ÁLGEBRA REAL

$$3x + 3x = 6x$$

$$2 + 3x = 2 + 3x$$

$$2 + 4x + 5 + x = 7 + 5x$$

ÁLGEBRA Nºˢ COMPLEXOS

$$3i + 3i = 6i$$

$$2 + 3i = 2 + 3i$$

$$2 + 4i + 5 + i = 7 + 5 \cdot i$$

$$z = 3 + 5i$$

$$w = -1 + 8i$$

$$z + w = (3 + (-1)) + (5 + 8) \cdot i$$

$$= 2 + 13 \cdot i$$

EXEMPLO

a) $z = 2 + i$ e $w = -1 - i$

i) $z + w = (2 + (-1)) + (1 + (-1)) \cdot i$

$= 1$ //

ii) $z - w = (2 - -1)) + (1 - (-1))\, i$

$= 3 + 2 \cdot i$

EXEMPLO

b) $z = -4 + 2i$ e $w = 1 - 2i$

i) $z - 2w = (-4 + 2i) - 2(1 - 2i)$

$= (-4 + 2i) + (-2 + 2i)$

$= -6 + 4i$

ii) $2z + w = 2 \cdot (-4 + 2i) + (1 - 2i)$

$= (-8 + 4i) + (1 - 2i)$

$= -7 + 2i$ //

PRODUTO DE NÚMEROS COMPLEXOS

Se $z = a + bi$ e $w = x + yi$

então $z \cdot w = (a + bi) \cdot (x + yi)$

$$= \underline{a \cdot x} + \underline{a \cdot y \cdot i} + \underline{b \cdot x \cdot i} + \underline{b \cdot y \cdot i^2}$$

$$= a \cdot x + b \cdot y(-1) + a \cdot y \cdot i + b \cdot x \cdot i$$

$$= (ax - by) + (a \cdot y + b \cdot x) \cdot i$$

EXEMPLO 01

Sejam os números complexos $z = 2 - i$ e $w = 3 + i$. Calcule $z \cdot w$ e $\frac{z}{w}$

R:

(i) $$z \cdot w = (2-i)\cdot(3+i) = 6 + 2i - 3i - i^2$$

$$= 6 - i + 1$$

$$= \underline{7 - i}$$

(ii) $$\frac{z}{w} = \frac{(2-i)}{(3+i)} \times \frac{(3-i)}{(3-i)} = \frac{6-2i-3i+i^2}{3^2 - i^2}$$

$$= \frac{5-5i}{10} = \frac{5}{10} - \frac{5}{10}i$$

$$= \frac{1}{2} - \frac{1}{2}i$$

EXEMPLO 02

Sejam os números complexos $z = 1 + i$ e $w = -1 + 2i$. Calcule $z \cdot w$ e $\dfrac{w}{z^6}$

R:

(i) $z \cdot w = (1+i)(-1+2i) = -1 + 2i - i \overbrace{+2i^2}^{-2}$

$$= -3 + i$$

(ii) $\dfrac{w}{z^6} = \dfrac{-1+2i}{-8i} \times \dfrac{i}{i} = \dfrac{-i+2i^2}{-8 \cdot i^2} = \dfrac{-2-i}{8}$

$$= -\frac{1}{4} - \frac{1}{8} i$$

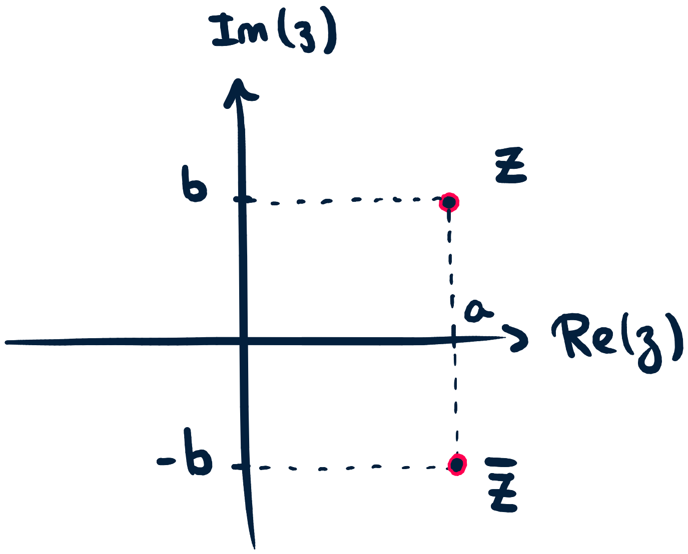
$z^2 = (1+i)^2 = \cancel{1} + 2 \cdot 1 \cdot i + \cancel{i^2} = 2i$

$z^6 = (z^2)^3 = (2i)^3 = 2^3 \cdot i^3 = 8(-i) = -8i$

CONJUGADO DE NÚMEROS COMPLEXOS

O conjugado de $z = a + bi$ é dado por

$$\boxed{\bar{z} = a - b \cdot i}$$



- z e $\bar{z}$ são simétricos em relação ao eixo real.

$z_1 = 2 + 3i \longrightarrow \overline{z_1} = 2 - 3i$

$z_2 = 5 - 2i \longrightarrow \overline{z_2} = 5 + 2i$

$z_3 = 2 \longrightarrow \overline{z_3} = 2$

$z_4 = 2 - i \longrightarrow \overline{z_4} = 2 + i$

$z_5 = 3 + 5i \longrightarrow \overline{z_5} = 3 - 5i$

$z_6 = 5i \longrightarrow \overline{z_6} = -5i$

EXEMPLO

Encontre as raizes da equação $x^2 - 4x + 5 = 0$

R:

$$\Delta = 16 - 4 \cdot 1 \cdot 5 = -4$$

$$\sqrt{\Delta} = \sqrt{-4} = \sqrt{4 \cdot (-1)} = \sqrt{4} \cdot \sqrt{-1} = 2 \cdot i$$

$$x = \frac{4 \pm 2i}{2 \cdot 1} = 2 \pm i$$

$$\boxed{x_1 = z = 2 + i}$$

$$\boxed{x_2 = \bar{z} = 2 - i}$$

PROPRIEDADES DO CONJUGADO

(i) $\overline{z+w} = \overline{z} + \overline{w}$

$z = a + bi \longrightarrow \overline{z} = a - bi$

$w = x + yi \longrightarrow \overline{w} = x - yi$

$z + w = (a+x) + (b+y)i$ $\qquad$ $\overline{z} + \overline{w} = (a+x) - (b+y).i$

$\overline{z+w} = (a+x) - (b+y)i$

(ii) $\overline{z-w} = \overline{z} - \overline{w}$

(iii) $\overline{z \cdot w} = \overline{z} \cdot \overline{w}$

$z = 2 + 3i$

$w = 1 - 2i$

$\overline{z} \cdot \overline{w} = (2-3i)(1+2i)$

$= 2 + 4i - 3i \underbrace{-6i^2}_{+6} = 8 + i$

$z \cdot w = (2+3i)(1-2i) = 2 - 4i + 3i \underbrace{-6i^2}_{+6} = 8 - i$

$\overline{z \cdot w} = 8 + i$

(iv) Se $w \neq 0$, $\overline{\left(\frac{z}{w}\right)} = \frac{\overline{z}}{\overline{w}}$

v) Se z é real então $\overline{z} = z$

↳ $z \in \mathbb{R}$: $z = a + 0 \cdot i = a$

$\overline{z} = a - 0 \cdot i = a$

(vi) $\overline{(\overline{z})} = z$

↳ $z = a + bi$

$\overline{z} = a - bi$

$\overline{(\overline{z})} = a + bi$

(vii) Se $n \in \mathbb{N}$: $\overline{z}^n = \overline{z^n}$

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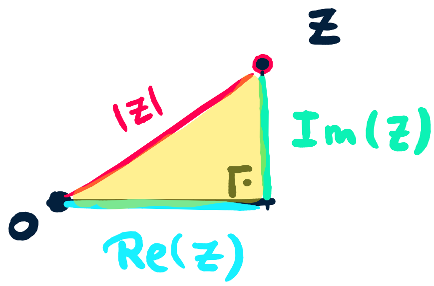
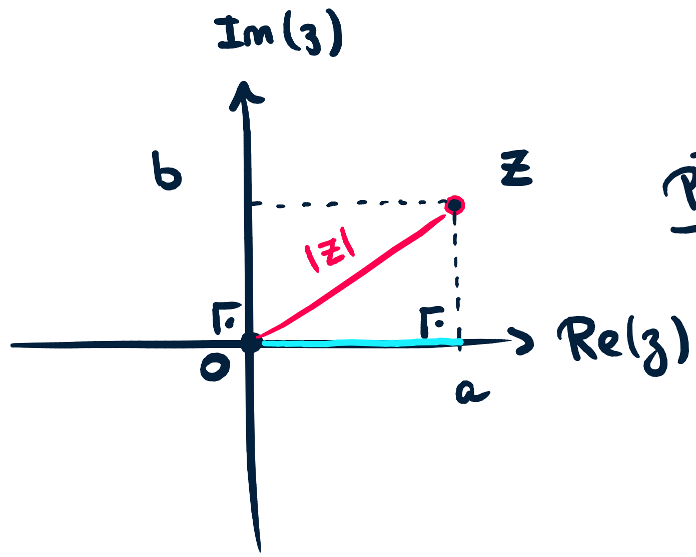
MÓDULO DE NÚMEROS COMPLEXOS

MEDE A DISTÂNCIA DO SEU AFIXO À ORIGEM





Pitágoras:

$$|z|^2 = [Re(z)]^2 + [Im(z)]^2$$

$$\boxed{|z| = \sqrt{[Re(z)]^2 + [Im(z)]^2}}$$

- $z_1 = 4 + 3i \longrightarrow |z_1| = \sqrt{4^2 + 3^2} = 5$

- $z_2 = -2 + i \longrightarrow |z_2| = \sqrt{(-2)^2 + 1^2} = \sqrt{5}$

- $z_3 = -4i \longrightarrow |z_3| = 4$

PROPRIEDADES DO MÓDULO

(i) $|z|^2 = z \cdot \bar{z}$

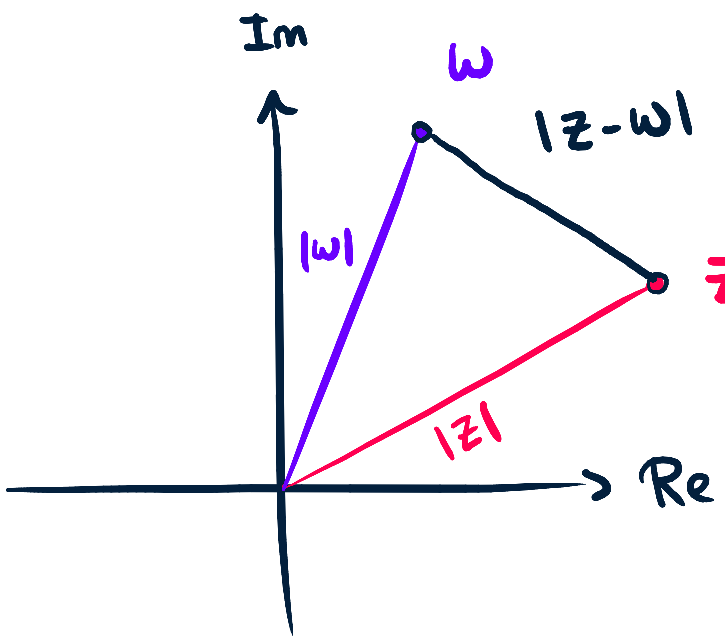
Seja $z = a + b \cdot i$ $\longrightarrow$ $|z|^2 = a^2 + b^2$

$\bar{z} = a - b \cdot i$

$$z \cdot \bar{z} = (a+bi) \cdot (a-bi) = a^2 - (bi)^2 = a^2 - b^2 \cdot i^2 = a^2 + b^2$$

(ii) A distância entre os complexos z e w

$d(z,w)$ é dada por $d(z,w) = |z - w|$

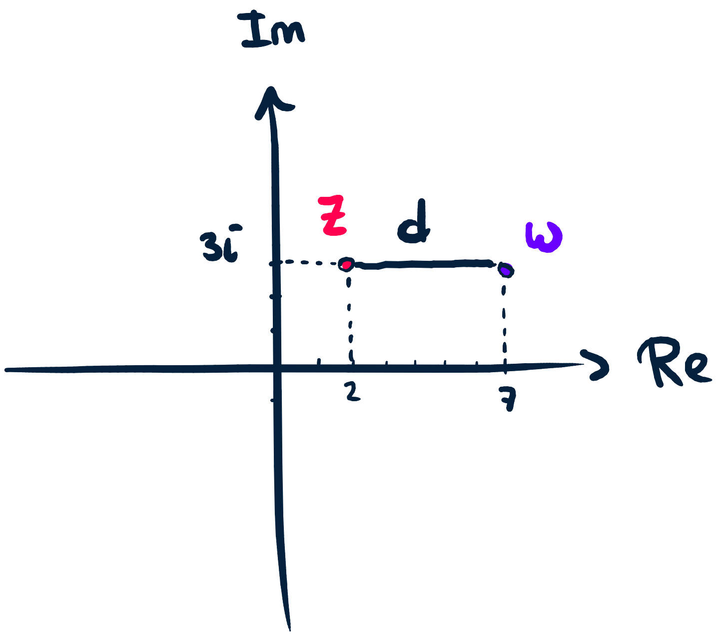


$|z| = |z - 0|$

$|w| = |w - 0|$

$z = 2 + 3i$

$w = 7 + 3i$



$d(z, w) = 5$

$|z - w| = |-5| = 5$

EXEMPLO 01

Escreva na forma algébrica o nº

$$z = \frac{(-1-i)^{80}}{i^{102}}$$

R:

$$(-1-i)^2 = [-(1+i)]^2 = (1+i)^2 = \cancel{1} + 2\cdot i + \cancel{i^2} = 2i$$

$$z = \frac{(-1-i)^{80}}{i^{102}} = \frac{\left[(-1-i)^2\right]^{40}}{i^{102}} = \frac{(2i)^{40}}{i^{102}}$$

$$= 2^{40}\cdot\frac{i^{40}}{i^{102}} = 2^{40}\cdot\frac{1}{-1}$$

$$= -2^{40}$$

$i^1 = i$

$i^2 = -1$

$i^3 = -i$

$i^4 = 1$

$Re(z) = -2^{40}$

$Im(z) = 0$

EXEMPLO 02

Determine $x \in \mathbb{R}$ para que o complexo

$$z = \frac{x+i}{2-i}$$ seja imaginário puro.

R:

$$z = \frac{(x+i)}{(2-i)} \cdot \frac{(2+i)}{(2+i)} = \frac{2x + x \cdot i + 2i + \overbrace{i^2}^{-1}}{2^2 - i^2}$$

$$= \frac{(2x-1) + (x+2) \cdot i}{5} = \frac{(2x-1)}{5} + \frac{(x+2)}{5} i$$

$$= \underbrace{a}_{\text{zero}} + b \cdot i$$

$$\frac{2x-1}{5} = 0 \therefore \boxed{x = 1/2}$$

$$\hookrightarrow z = \frac{(x+2)}{5} i = \frac{1}{2} i$$

EXEMPLO 03

Determine $z \in \mathbb{C}$ tal que $|z| = 5 \cdot z \cdot i$

R :

Seja $z = a + bi$

$|z|^2 = 5^2 \cdot z^2 \cdot i^2 \therefore |z|^2 = -25 z^2$

$a^2 + b^2 = -25 \underbrace{(a + bi)^2}$

$a^2 + 2abi + (bi)^2 = a^2 - b^2 + 2ab \cdot i$

$a^2 + b^2 = -25(a^2 - b^2 + 2abi) = -25(a^2 - b^2) - 50abi$

$a^2 + b^2 + 25a^2 - 25b^2 + 50abi = 0$

$w = (\underbrace{26a^2 - 24b^2}_{\text{zero}}) + (\underbrace{50 \cdot a \cdot b}_{\text{zero}})i = 0$

$\left.\begin{array}{l} i)\ 50ab = 0 \longrightarrow a = 0 \\ ii)\ 26\cancel{a^2}^{0} - 24b^2 = 0 \longrightarrow b = 0 \end{array}\right\} \begin{array}{l} z = a + bi \\ = 0 \end{array}$

EXEMPLO 04

Determine as soluções de $z^2 + |z| = 0$

R:

(i) $z = a + bi$

$$z^2 + |z| = 0 \therefore (a+bi)^2 + \sqrt{a^2+b^2} = 0$$

$$a^2 + 2abi + b^2i^2 = -\sqrt{a^2+b^2}$$

$$\left[(a^2-b^2) + 2abi\right]^2 = \left[-\sqrt{a^2+b^2}\right]^2$$

$$(a^2-b^2)^2 + 2(a^2-b^2)\cdot 2abi - 4a^2b^2 = a^2+b^2$$

$$\left[(a^2-b^2)^2 - (a^2+4a^2b^2+b^2)\right] + \left[4ab(a^2-b^2)\right]\cdot i = 0$$

i) $Im = 0 \therefore 4ab(a^2-b^2) = 0$

1) $a = 0$
2) $b = 0$
3) $a = b$
4) $a = -b$

ii) $Re = 0 \therefore (a^2-b^2)^2 - (a^2+4a^2b^2+b^2) = 0$

1) $a = 0$

$$(\cancel{a^2}_{0} - b^2)^2 - (\cancel{a^2}_{0} + 4\cancel{a^2}_{0} b^2 + b^2) = 0 \quad \therefore \quad b^4 - b^2 = 0$$

$$b^2 (b^2 - 1) = 0$$

$b = 0$ $\qquad$ $b = 1$, $b = -1$

<u>Soluções</u>: $(a, b) = (0, 0) : 0$

$(0, 1) : 0 + i$

$(0, -1) : 0 - i$

2) $b = 0$

$$(a^2 - \cancel{b^2}_{0})^2 - (a^2 + 4\cancel{a^2 b^2}^{0} + \cancel{b^2}^{0}) = 0 \quad \therefore \quad a^4 - a^2 = 0$$

$a = 0$ ou $a = \pm 1$

<u>Soluções</u>: $(a, b) = (0, 0), (1, 0), (-1, 0)$
$\qquad\qquad\qquad\quad 0 \quad , \quad 1 + 0i \quad , \quad -1 + 0i$

3 e 4) $a = \pm b$

$$(\underbrace{a^2 - b^2}_{\text{zero}})^2 - (a^2 + 4a^2 b^2 + b^2) = 0 \quad \therefore \quad (a^2 + 4a^2 . a^2 + a^2) = 0$$

$$2a^2 + 4a^4 = 0 \Rightarrow a = 0 \Rightarrow b = 0 : (a, b) = (0, 0)$$

<u>Solução</u>: $z = \{ 0, \cancel{1}, -\cancel{1}, i, -i \}$

✓ ✓ ✓

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EXEMPLO 05 (ITA 1993)

Encontre as soluções de $z^2 = \overline{2+z}$

R: $z = a + bi$, $a \in \mathbb{R}$, $b \in \mathbb{R}$

$$z^2 = (a+bi)^2 = a^2 + 2abi - b^2$$

$$\overline{2+z} = \overline{(2+a+bi)} = (2+a-bi)$$

$$z^2 = \overline{2+z} \therefore a^2 + 2abi - b^2 = 2 + a - bi$$

$$\underbrace{(a^2 - b^2 - a - 2)}_{Re(w)} + \underbrace{(2ab+b)\cdot i}_{Im(w)} = 0$$

i) $Im(w) = 0 \therefore b(2a+1) = 0$ → $\boxed{b = 0}$ ou $\boxed{a = -1/2}$

ii) $Re(w) = 0 \therefore a^2 - b^2 - a - 2 = 0$

OP. A: $a = -1/2 \Rightarrow b \in \mathbb{C}$ ✗

OP. B: $b = 0$ → $a = 2$ → $a = -1$

$$\boxed{\begin{array}{c} z_1 = 2 \\ \text{ou} \\ z_2 = -1 \end{array}}$$

EXEMPLO 06 (ITA 1987)

Seja $u = z \cdot w + \overline{z \cdot w}$, com $z, w \in \mathbb{C}$.

$\bar{u}$ é igual a

a) $|z| \cdot |w|$

b) imaginário puro

c) $2\,Re(z+w)$

d) $2\,Re(z \cdot w)$

R: $z = a + b \cdot i$ e $w = x + yi$

$\cdot\ z \cdot w = (a+bi)(x+yi) = a.x + a.y.i + b.x.i - by$

$= (ax - by) + (a.y + bx)\,i$

$\cdot\ \overline{zw} = (ax - by) - (a.y + bx)\,i$

$u = z \cdot w + \overline{z \cdot w} = (ax - by) + \cancel{(a.y + bx)\,i} +$

$(ax - by) - \cancel{(a.y + bx)\,i}$

$u = 2(ax - by) = 2Re(u)$

$\bar{u} = u$